DETERMINATION OF THE ABSTRACT GROUPS OF ORDER

$16\ p^2$ and $8\ p^3$

INAUGURAL DISSERTATION

BY

RAGNAR NYHLÉN
LIC. PHIL., VÄRML.

BY DUE PERMISSION OF THE PHILOSOPHICAL FACULTY (NATURAL SCIENCE SECTION) OF THE UNIVERSITY OF UPSALA TO BE PUBLICLY DISCUSSED IN LECTURE ROOM XI, DECEMBER 6th, 1919, AT 10 O'CLOCK A. M. FOR THE DEGREE OF DOCTOR OF PHILOSOPHY

UPPSALA 1919
APPELBERGS BOKTRYCKERI AKTIEBOLAG

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Introduction.

The problem of finding all the types of non-isomorphic groups of a given order was formulated by Cayley. The groups are to be conceived as generated by certain fundamental operations connected by a number of independent relations. For the lowest numbers ($n \leq 32$) the groups were determined by Cayley, Kempe, Miller and Burnside (*1 — 4*)[1] Kempe gave also a graphic representation of all G_n ($n \leq 12$). The problem was dealt with more generally by Netto (*5*), who gave all the groups G_{p^2} and G_{pq}. He did not, however, indicate any general method. Hölder was the first to determine in a treatise (*6*) more general methods for the building-up of compound groups from given factor-groups. In the treatise mentioned he also solved the problem for p^3, pq^2, pqr and p^4. Young, Cole and Glover also dealt with the same types of groups about the same time as Hölder (*7*, *8*). Groups whose order contains a smaller number of prime factors — equal or unequal — are investigated below and the respective groups have been set up (*9 — 19*). When the number of prime factors entering into the order of the group is increased, the problem soon becomes remarkably complicated. The present treatise aims at giving the complete solution of the problem in respect of groups whose order is $16p^2$ and $8p^3$. From the defining relations set up, certain characteristics of the groups investigated may be discovered. Some groups are, e. g., the direct product of two or more groups of lower order or may be determined by n — to — n' isomorphism in two groups of lower order. The solubility of the groups $G_{p^\alpha q^\beta}$ has been proved for all α and β by Burnside (*20*).

[1] The figures in italics refer to corresponding figures in the Bibliography.

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The groups of order $16p^2$.

The various groups of order 16[1] appear here as sub-groups. As a knowledge of their self-conjugate sub-groups simplifies the investigation considerably these groups are given here.

$$G_{16}^1 = \{A^{16} = 1\}$$

The sub-groups are $\{A^2\}$, $\{A^4\}$ and $\{A^8\}$. The order[2] of the group of isomorphisms is 8.

$$G_{16}^2 = \{A^8 = B^2 = 1 \quad AB = BA\}$$

The sub-groups are $\{A\}$, $\{AB\}$, $\{A^2, B\}$ of order 8, $\{A^2\}$, $\{A^2B\}$, $\{A^4, B\}$ of order 4 and $\{A^4\}$, $\{B\}$, $\{A^4B\}$ of order 2. The order of the group of isomorphisms is 16.

$$G_{16}^3 = \{A^4 = B^4 = 1 \quad AB = BA\}$$

The sub-groups are $\{A, B^2\}$, $\{A^2, B\}$, $\{AB, A^2\}$ of order 8, $\{A\}$, $\{B\}$, $\{AB\}$, $\{AB^2\}$, $\{AB^3\}$, $\{A^2B\}$, $\{A^2, B^2\}$ of order 4 and $\{A^2\}$, $\{B^2\}$, $\{A^2B^2\}$ of order 2. The order of the group of isomorphisms is 96.

$$G_{16}^4 = \{A^4 = B^2 = C^2 = 1 \quad AB = BA \ \ AC = CA \ \ BC = CB\}$$

The sub-groups are $\{A^2, B, C\}$, $\{A, B\}$, $\{A, C\}$, $\{A, BC\}$, $\{AB, C\}$, $\{AB, BC\}$, $\{AC, B\}$ of order 8, $\{A\}$, $\{AB\}$, $\{AC\}$, $\{ABC\}$, $\{A^2, B\}$, $\{A^2, C\}$, $\{A^2, BC\}$, $\{B, C\}$, $\{B, A^2C\}$, $\{A^2B, C\}$, $\{A^2B, A^2C\}$ of order 4 and $\{A^2\}$, $\{B\}$, $\{A^2B\}$, $\{C\}$, $\{A^2C\}$, $\{BC\}$ $\{A^2BC\}$ of order 2. The order of the group of isomorphisms is 192.

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[1] Burnside, Theory of groups of finite order. Cap. 10 (1911).
[2] Levavasseur, Les groups d'ordre $16p$. Toulouse Ann. (1903).

$$G_{16}^{5} = \begin{cases} A^2 = B^2 = C^2 = D^2 = 1 \quad AB = BA \quad AC = CA \\ AD = DA \quad BC = CB \quad BD = DB \quad CD = DC \end{cases}$$

There are 15 sub-groups of the type $\{A, B, C\}$, 35 of the type $\{A, B\}$ and 15 of order 2. The order of the group of isomorphisms is 20160.

$$G_{16}^{6} = \{A^8 = B^2 = 1 \quad B^{-1}AB = A^5\}$$

The self-conjugate sub-groups are $\{A\}$, $\{BA\}$, $\{A^2, B\}$ (Abelian) of order 8, $\{A^2\}$, $\{BA^2\}$, $\{A^4, B\}$ of order 4 and $\{A^4\}$ of order 2. The order of the group of isomorphisms is 16.

$$G_{16}^{7} = \begin{cases} A^2 = B^2 = C^4 = 1 \quad B^{-1}AB = AC^2 \\ AC = CA \quad BC = CB \end{cases}$$

The self-conjugate sub-groups are $\{C, A\}$, $\{C, B\}$, $\{C, BAC\}$ (Abelian), $\{AC, B\}$, $\{BC, A\}$, $\{BA, B\}$ (dihedral groups), $\{AC, BC\}$ (quaternion-group) of order 8, $\{C\}$, $\{AC\}$, $\{BC\}$, $\{BA\}$, $\{C^2, A\}$, $\{C^2, B\}$, $\{C^2, BAC\}$ of order 4 and $\{C^2\}$ of order 2. The order of the group of isomorphisms is 48.

$$G_{16}^{8} = \{A^4 = B^4 = 1 \quad B^{-1}AB = A^3\}$$

The self-conjugate sub-groups are $\{A, B^2\}$, $\{B, A^2\}$, $\{BA, A^2\}$ (Abelian) of order 8, $\{A\}$, $\{B^2A\}$, $\{A^2, B^2\}$ of order 4 and $\{A^2\}$, $\{B^2\}$, $\{A^2B^2\}$ of order 2. The order of the group of isomorphisms is 32.

$$G_{16}^{9} = \begin{cases} A^4 = B^2 = C^2 = 1 \quad B^{-1}AB = A^3 \\ BC = CB \quad AC = CA \end{cases}$$

The self-conjugate sub-groups are $\{A^2, B, C\}$, $\{A^2, BA, C\}$, $\{A, C\}$ (Abelian), $\{A, B\}$, $\{A, BC\}$, $\{AC, B\}$, $\{AC, BA\}$ (dihedral groups) of order 8, $\{A\}$, $\{AC\}$, $\{A^2, B\}$, $\{A^2, AB\}$, $\{A^2, BC\}$, $\{A^2, BAC\}$, $\{A^2, C\}$ of order 4 and $\{A^2\}$, $\{C\}$, $\{A^2C\}$ of order 2. The order of the group of isomorphisms is 64.

$$G_{16}^{10} = \begin{cases} A^4 = B^2 = C^2 = 1 \quad B^{-1}AB = AC \\ AC = CA \quad BC = CB \end{cases}$$

The self-conjugate sub-groups are $\{A^2, B, C\}$, $\{A, C\}$, $\{AB, A^2\}$ (Abelian) of order 8, $\{A^2, C\}$, $\{B, C\}$, $\{A^2B, C\}$ of order 4 and $\{A^2\}$, $\{A^2C\}$, $\{C\}$ of order 2. The order of the group of isomorphisms is 32.

$$G_{16}^{11} = \begin{cases} A^4 = B^4 = C^2 = 1 & B^{-1}AB = A^3 \\ A^2 = B^2 \quad AC = CA & BC = CB \end{cases}$$

The self-conjugate sub-groups are $\{A, C\}$, $\{B, C\}$, $\{AB, C\}$ (Abelian) $\{A, B\}$, $\{A, BC\}$, $\{B, AC\}$, $\{AB, AC\}$ (quaternion-groups) of order 8, $\{A\}$, $\{B\}$, $\{AB\}$, $\{AC\}$, $\{BC\}$, $\{ABC\}$, $\{A^2, C\}$ of order 4 and $\{A^2\}$, $\{C\}$, $\{A^2C\}$ of order 2. The order of the group of isomorphisms is 192.

$$G_{16}^{12} = \{A^8 = B^2 = 1 \quad B^{-1}AB = A^{-1}\}$$

The self-conjugate sub-groups are $\{A\}$ (Abelian), $\{A^2, B\}$, $\{A^2, BA\}$ (dihedral groups) of order 8, $\{A^2\}$ of order 4 and $\{A^4\}$ of order 2. The order of the group of isomorphisms is 32.

$$G_{16}^{13} = \{A^8 = B^2 = 1 \quad B^{-1}AB = A^3\}$$

The self-conjugate sub-groups are $\{A\}$ (Abelian), $\{A^2, B\}$ (dihedral group), $\{A^2, BA\}$ (quaternion-group) of order 8, $\{A^2\}$ of order 4 and $\{A^4\}$ of order 2. The order of the group of isomorphisms is 16.

$$G_{16}^{14} = \{A^8 = B^4 = 1 \quad A^4 = B^2 \quad B^{-1}AB = A^{-1}\}$$

The self-conjugate sub-groups are $\{A\}$ (Abelian), $\{A^2, B\}$, $\{A^2, AB\}$ (quaternion-groups) of order 8, $\{A^2\}$ of order 4 and $\{A^4\}$ of order 2. The order of the group of isomorphisms is 32.

Sylow's theorem proves that, with the exception of $p = 3$, 5 and 7, all the investigated groups G_{16p^2} contain a self-conjugate G_{p^2}.

I.

The G_{16p^2} which have a self-conjugate G_{16} and also a self-conjugate G_{p^2}.

The sub-groups G_{16} and G_{p^2} have no common operation besides the identity. Every operation in G_{16} must therefore be permutable with every operation in G_{p^2}. The investigated

G_{16p^2} are thus obtained as the direct product of 1 G_{16} and 1 G_{p^2}. 28 types are obtained, 10 of which are Abelian. The sub-groups of the direct product of 1 G_{16}^{11} and 1 G_{p^2} are all self-conjugate. These two groups are the only G_{16p^2} besides the Abelian ones possessing this property.

II.

The G_{16p^2} which contain a self-conjugate cyclic G_{p^2} and more than 1 G_{16}.

The factor-group G_{16p^2}/G_{p^2} is isomorphic with any one of the 14 types G_{16}. The group of isomorphisms of G_{p^2} is a cyclic group of order $p\,(p-1)$. If no operation in G_{16} is permutable with every operation in G_{p^2} then G_{16} must be simply isomorphic with the group of isomorphisms of G_{p^2} or one of its sub-groups. G_{16} must consequently be cyclic. The operations in G_{16} which are permutable with every operation in G_{p^2} form a self-conjugate sub-group H of G_{16}. To every operation of the factor-group G_{16}/H there corresponds an isomorphism of G_{p^2}, wherefore this factor-group must be cyclic. Two sub-groups H, one of which is transferred to the other by taking new generating operations of G_{16}, give isomorphic groups G_{16p^2}.

The result is set forth in the following table:

G_{16p^2}/G_{p^2}	H	$A^{-1}PA$	$B^{-1}PB$	$C^{-1}PC$	$D^{-1}PD$	*Aritm. Rel.*
G_{16}^{1}	$\{A^2\}$	P^{-1}				
	$\{A^4\{$	P^{α}				$p\equiv 1 (\mathrm{mod}\ 4)$
	$\{A^8\}$	P^{α}				$p\equiv 1 (\mathrm{mod}\ 8)$
	E	P^{α}				$p\equiv 1 (\mathrm{mod} 16)$
G_{16}^{2}	$\{A\}$	P	P^{-1}			
	$\{A^2, B\}$	P^{-1}	P			
	$\{A^4, B\}$	P^{α}	P			$p\equiv 1 (\mathrm{mod}\ 4)$
	$\{BA^2\}$	P^{α}	P^{-1}			$p\equiv 1 (\mathrm{mod}\ 4)$
	B	P^{α}	P			$p\equiv 1 (\mathrm{mod}\ 8)$

G_{16p^2}/G_{p^2}	H	$A^{-1}PA$	$B^{-1}PB$	$C^{-1}PC$	$D^{-1}PD$	*Aritm. Rel.*
G^{3}_{16}	$\{A, B^2\}$	P	P^{-1}			
	$\{A\}$	P	P^{α}			$p \equiv 1 \pmod 4$
G^{4}_{16}	$\{A^2, B, C\}$	P^{-1}	P	P		
	$\{A, B\}$	P	P	P^{-1}		
	$\{B, C\}$	P^{α}	P	P		$p \equiv 1 \pmod 4$
G^{5}_{16}	$\{A, B, C\}$	P	P	P	P^{-1}	
G^{6}_{16}	A	P	P^{-1}			
	$\{A^2, B\}$	P^{-1}	P			
	$\{A^4, B\}$	P^{α}	P			$p \equiv 1 \pmod 4$
	$\{BA^2\}$	P^{α}	P^{-1}			$p \equiv 1 \pmod 4$
G^{7}_{16}	$\{A, C\}$	P	P^{-1}	P		
	$\{AC, B\}$	P^{-1}	P	P^{-1}		
	$\{AC, BC\}$	P^{-1}	P^{-1}	P^{-1}		
G^{8}_{16}	$\{A, B^2\}$	P	P^{-1}			
	$\{A^2, B\}$	P^{-1}	P			
	$\{A\}$	P	P^{α}			$p \equiv 1 \pmod 4$
G^{9}_{16}	$\{A^2, B, C\}$	P^{-1}	P	P		
	$\{A, C\}$	P	P^{-1}	P		
	$\{A, B\}$	P	P	P^{-1}		
G^{10}_{16}	$\{A^2, B, C\}$	P^{-1}	P	P		
	$\{A, C\}$	P	P^{-1}	P		
	$\{B, C\}$	P^{α}	P	P		$p \equiv 1 \pmod 4$
G^{11}_{16}	$\{A, B\}$	P	P	P^{-1}		
	$\{A, C\}$	P	P^{-1}	P		
G^{12}_{16}	$\{A\}$	P	P^{-1}			
	$\{A^2, B\}$	P^{-1}	P			
G^{13}_{16}	$\{A\}$	P	P^{-1}			
	$\{A^2, B\}$	P^{-1}	P			
	$\{A^2, BA\}$	P^{-1}	P^{-1}			
G^{14}_{16}	$\{A\}$	P	P^{-1}			
	$\{A^2, B\}$	P^{-1}	P			

In every case a fixed value can be taken for a. The remaining a-values give isomorphic groups with this one. Morover all the groups obtained are distinct, as appears from the method of arrangement.

III.

The G_{16p^2} which contain a self-conjugate non-cyclic G_{p^2} and more than 1 G_{16}.

The factor-group G_{16p^2}/G_{p^2} is isomorphic with any one of the 14 types G_{16}. The operations in G_{16} which are permutable with every operation in G_{p^2} form a self-conjugate sub-group H of G_{16}. To every operation in G_{16}/H there then corresponds an isomorphism of G_{p^2}. The arrangement of the various G_{16p^2} is decided by the type of H, on account of which this is considerably simplified. Two groups H which cannot be transferred the one to the other by altering the generating operations of G_{16} give distinct groups.

G_{16}/H being cyclic, the results are obtained directly, as the G_{16p^2} which have a self-conjugate G_{p^2} and more than a cyclic G_{16} have been previously produced.

$$G_{16p^2}/G_{p^2} = \{A^{16} = 1\}\,[1]$$

We have
$$A^{-1}P_1A = P_1{}^{\alpha}P_2{}^{\beta}$$
$$A^{-1}P_2A = P_1{}^{\gamma}P_2{}^{\delta}$$

whence it follows that
$$A^{-1}P_1{}^{x}P_2{}^{y}A = P_1{}^{\alpha x + \gamma y}P_2{}^{\beta x + \delta y}$$

If $\alpha\delta - \beta\gamma \not\equiv 0 \pmod p$ this gives an isomorphism of G_{p^2}. In order that there may be in G_{p^2} sub-groups of order p with which A is permutable it is necessary that

$$\sigma^2 - \sigma\,(\alpha + \delta) + \alpha\delta - \beta\gamma \equiv 0 \pmod p \quad . \quad . \quad . \quad (1)$$

If σ is a solution of this congruence x and y can be found, so that

$$A^{-1}P_1{}^{x}P_2{}^{y}A = (P_1{}^{x}P_2{}^{y})^{\sigma}$$

[1] This expression indicates that the factor-group G_{16p^2}/G_{p^2} is isomorphic with a cyclic G_{16}.

The congruence (1) is reducible

G_{p^2} contains $p+1 G_p$, which are permuted when transformed by A. Of these l are each permutable with A. The rest are permuted in cycles with 2, 4, 8 or 16 in each

$$\because\ p+1 = l + 2m,$$

whence it follows that $l \neq 1$.

At least two sub-groups G_p are each permutable with A. For this $\{P_1\}$ and $\{P_2\}$ may be chosen

$$\because\ A^{-1}P_1A = P_1^{\alpha} \quad A^{-1}P_2A = P_2^{\beta}$$

$$\because\ \alpha^{16} \equiv \beta^{16} \equiv 1 \pmod{p}$$

$H = \{A^2\}$ $\alpha \equiv -1$ $\beta \equiv \pm 1$ then give two distinct types.

$H = \{A^4\}$ α can be chosen as a fixed primitive root of $\alpha^4 \equiv 1$. The four different values of β all give distinct types.

$H = \{A^8\}$ α a fixed primitive root of $\alpha^8 \equiv 1$. The eight different values of β all give distinct types.

$H = E$ α a fixed primitive root of $\alpha^{16} \equiv 1$. The fourteen different values $\beta = \alpha^r$ $(r \neq 11,\ 13)$ all give distinct types.

The congruence (1) is irreducible.

P_1 and P_2 can be chosen so that they are conjugated when transformed by A

$$\because\ J_A = (P_1,\ P_2;\ P_2,\ P_1^{\gamma}P_2^{\delta})$$ [1]

This isomorphism may be of order 4, 8 or 16.

$$J_{A^2} = (P_1,\ P_2;\ P_1^{\gamma}P_2^{\delta},\ P_1^{\gamma\delta}P_2^{\gamma+\delta^2})$$

$$J_{A^4} = (P_1,\ P_2;\ P_1^{\gamma(\gamma+\delta^2)}P_2^{\delta(2\gamma+\delta^2)},\ P_1^{\gamma\delta(2\gamma+\delta^2)}P_2^{3\gamma\delta^2+\gamma^2+\delta^4})$$

$$J_{A^8} = (P_1,\ P_2;\ P_1^{\gamma^2(\gamma+\delta^2)^2+\gamma\delta^2(2\gamma+\delta^2)^2}P_2^{\delta(2\gamma+\delta^2)\ (4\gamma\delta^2+2\gamma^2+\delta^4)},$$

$$P_1^{\gamma^2\delta(\gamma+\delta^2)\ (2\gamma+\delta^2)+\gamma\delta(3\gamma\delta^2+\gamma^2+\delta^4}P_2^{\gamma\delta^2(2\gamma+\delta^2)^2+(3\gamma\delta^2+\gamma^2+\delta^4)^2}$$

In order that A^8 may be permutable with P_1 it is necessary that

$$\left.\begin{aligned} \delta(2\gamma+\delta^2)\ (4\gamma\delta^2+2\gamma^2+\delta^4) &\equiv 0 \\ \gamma^2(\gamma+\delta^2)^2+\gamma\delta^2\ (2\gamma+\delta^2)^2 &\equiv 1 \end{aligned}\right\} \pmod{p}$$

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[1] For the sake of brevity the above sign is employed for the isomorphism.

$\therefore\ \delta$ or $2\gamma + \delta^2$ or $4\gamma\delta^2 + 2\gamma^2 + \delta^4 \equiv 0 \pmod{p}$

(*i*) $\delta \equiv 0$ $\qquad \therefore\ \gamma^4 \equiv 1 \pmod{p}$

Supposing $\gamma \equiv 1$, the congruence is soluble for all p. If on the other hand $\gamma \equiv -1$, this congruence lacks solutions for $p \equiv 3 \pmod{4}$, which consequently gives a new type[1]. If γ belongs to the exponent 4 $(\mathrm{mod}\ p)$, then $p \equiv 1 \pmod{4}$. The congruence (1) lacks solutions for $p \equiv 5 \pmod{8}$. The two primitive roots give isomorphic types.

(*ii*) $2\gamma + \delta^2 \equiv 0$ $\qquad \therefore\ \gamma^4 \equiv 1 \pmod{p}$

Supposing $\gamma \equiv 1$, it follows that -2 is a quadratic remainder of p which occurs only for $p \equiv 1$ or $3 \pmod{8}$. The congruence (1) has no solutions for $p \equiv 3 \pmod{8}$. The two values of δ give isomorphic types. If on the other hand $\gamma \equiv -1$ $p \equiv \pm 1 \pmod{8}$ is necessary. The congruence (1) lacks solutions only for $p \equiv -1 \pmod{8}$. The two roots of $\delta^2 \equiv 2 \pmod{p}$ give only one distinct type. If γ belongs to the exponent 4 $(\mathrm{mod}\ p)$ $\delta^2 \equiv 2\gamma \pmod{p}$ is soluble for $p \equiv 1 \pmod{4}$, for which p-values the congruence (1) is soluble

(*iii*) $4\gamma\delta^2 + 2\gamma^2 + \delta^4 \equiv 0$ $\quad \therefore\ \gamma^4 \equiv -1 \pmod{p}$

γ therefore belongs to the exponent 8 $(\mathrm{mod}\ p)$, wherefore $p \equiv 1 \pmod{8}$. The congruence (1) will be in this case

$$\sigma^2 - \delta\sigma - \gamma \equiv 0$$

$$\therefore\ (2\sigma - \delta)^2 \equiv \delta^2 + 4\gamma, \text{ which has solutions}$$

if

$$(\delta^2 + 4\gamma)^{\frac{p-1}{2}} \equiv 1.$$

From (*iii*) we get $\qquad \delta^2(4\gamma + \delta^2) \equiv -2\gamma^2$

$$\therefore\ (4\gamma + \delta^2)^{\frac{p-1}{2}} \equiv 1 \qquad \text{for } p \equiv 1 \pmod{8}$$

Consequently no such group exists.

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[1] If we suppose $p \equiv 1 \pmod{4}$ these defining relations give a type isomorphic with $\{A,\ P_1,\ P_2\}$ where $J_A = (P_1,\ P_2;\ P_1^{\alpha},\ P_2^{\alpha^3})$. It is, however, simplest to separate these types below. Similar conditions can be proved for the other groups belonging here.

Suppose now that A^{16} is the lowest power of A which is permutable with P_1

If $$A^{-8}P_1A^8 = P_2$$

it follows that $$A^{-16}P_1A^{16} = A^{-8}P_2A^8 = P_1$$

and thus A^8 is permutable with P_1P_2.

Thus we get $A^{-8}P_1A^8 = P_1^{-1}$, whence it follows that

when $$A^{-1}P_2A = P_2 \quad A^{-1}P_2A = P_1^{\gamma}P_2^{\delta}$$

$$\left.\begin{aligned} \delta(2\gamma+\delta^2)\ (4\gamma\delta^2+2\gamma^2+\delta^4) &\equiv 0 \\ \gamma^2(\gamma+\delta^2)^2+\gamma\delta^2\ (2\gamma+\delta^2)^2 &\equiv -1 \end{aligned}\right\} \pmod p$$

$$\therefore\ \delta \quad \text{or} \quad 2\gamma+\delta^2 \quad \text{or} \quad 4\gamma\delta^2+2\gamma^2+\delta^4 \equiv 0 \pmod p$$

(*i*) $\delta \equiv 0$ $\qquad\qquad$ $\therefore\ \gamma^4 \equiv -1 \pmod p$.

γ thus belongs to the exponent 8 (mod p) wherefore $p \equiv 1$ (mod 8). The congruence (1) is irreducible for $p \equiv 9$ (mod 16). The four primitive roots of $\gamma^8 \equiv 1$ give isomorphic types

(*ii*) $2\gamma+\delta^2 \equiv 0$ $\qquad\qquad$ $\therefore\ \gamma^4 \equiv -1 \pmod p$

The congruence (1) is irreducible for $p \equiv 9$ (mod 16), but for prime numbers of this form $\delta^2 \equiv -2\gamma$ has no solution

(*iii*) $4\gamma\delta^2+2\gamma^2+\delta^4 \equiv 0$ $\qquad$ $\therefore\ \gamma^4 \equiv 1 \pmod p$

The congruence (1) becomes in this case

$$\sigma^2-\sigma\delta-\gamma \equiv 0$$

$$\therefore\ (2\sigma-\delta)^2 \equiv 4\gamma+\delta^2,$$

which is irreducible only when $4\gamma+\delta^2$ is a quadratic non-remainder of p. From (*iii*) we get

$$\delta^2(4\gamma+\delta^2) \equiv -2\gamma^2$$

Hence it follows that

$$(4\gamma+\delta^2)^{\frac{p-1}{2}} \equiv -1 \text{ for } p \equiv 5 \text{ or } 7 \pmod 8$$

Supposing $\gamma \equiv 1$, we get

$$(\delta^2+2)^2 \equiv 2 \pmod p$$

This congruence has no solutions for $p \equiv 15 \pmod{16}$, but has four solutions for $p \equiv 7 \pmod{16}$. We get a type for which $J_A = (P_1, P_2; P_2, P_1P_2^{\delta})$. The three remaining values of δ give groups which are isomorphic with the foregoing type. To show this we may, e. g., let $\{A, P_1, P_2\}$ be generated by $\{A^3, P_1, P_1^{\delta}P_2^{1+\delta^2}\}$, $\{A^9, P_1, P_2^{-1}\}$ or $\{A^{11}, P_1, P_1^{-\delta}P_2^{-1-\delta^2}\}$.

If on the other hand $\gamma \equiv -1$, we obtain for $p \equiv 15 \pmod{16}$ in the same way only one type for which $J_A = (P_1, P_2; P_2, P_1^{-1}P_2^{\delta})$.

As γ belongs to the exponent 4 (mod p), it is necessary that $p \equiv 5 \pmod 8$, for which p-values $(\delta^2 + 2\gamma)^2 \equiv 2\gamma^2$ has no solutions. Consequently there is no group belonging here.

$$G_{16p^2}/G_{p^2} = \{A^8 = B^2 = 1 \quad AB = BA\}$$

$$H = \{A\}$$

G^2_{16}/H is isomorphic with a G_2, which gives (Page 9) two types for which $J_B = (P_1, P_2; P_1^{-1}, P_2^{\pm 1})$.

When G_{16}/H is isomorphic with a G_2 we always get two distinct types.

$$H = \{A^2, B\} \because J_A = (P_1, P_2; P_1^{-1}, P_2^{\pm 1})$$

$$H = \{A^2\}$$

G^2_{16}/H is isomorphic with a non-cyclic G_4. $G_{8p^2} = \{A, P_1, P_2\}$ is a self-conjugate sub-group of G_{16p^2} where $J_A = (P_1, P_2; P_1^{\pm 1}, P_2^{-1})$ may be supposed without limitation. J_B, which is a general isomorphism of order 2 to G_{p^2}, must be permutable with J_A.

We get $$J_B = (P_1, P_2; P_1^a P^b, P_1^c P_2^d)$$

Because $J_B^2 = 1$ it follows

$$\left.\begin{array}{ll} a^2 + bc \equiv 1 & b(a+d) \equiv 0 \\ d^2 + bc \equiv 1 & c(a+d) \equiv 0 \end{array}\right\} \pmod p \quad . \quad . \quad (2)$$

$J_A = (P_1, P_2; P_1, P_2^{-1})$ is permutable with J_B

if $$b \equiv c \equiv 0$$

Hence it follows that $$a^2 \equiv d^2 \equiv 1$$

The four solutions give only two possible values for J_B. The two types obtained for which $J_B = (P_1, P_2; P_1^{-1} P_2^{\pm 1})$ are distinct.

$J_A = (P_1, P_2; P_1^{-1}, P_2^{-1})$ is a self-conjugate operation in the whole group of isomorphisms of G_{p^2}. J_B can thus be any isomorphism when a, b, c and d satisfy (2)

The congruences (2) give $a^2 \equiv d^2$

(*i*) $a \equiv d$ $\quad\because\ 2ab \equiv 2ac \equiv 0$

When $a \equiv 0$ it follows that

$$b \equiv c \equiv 0$$

$\because\ a^2 \equiv 1$, which gives no type belonging here

(*ii*) $a \equiv -d$ $\quad\because\ a^2 + bc \equiv 1$

J_B transforms two sub-groups of order p of G_{p^2} into themselves if

$$\sigma^2 - \sigma(a + d) + ad - bc \equiv 0 \pmod{p}$$

is reducible. From (*ii*) we get $\sigma^2 \equiv 1$, and consequently we can in this case always choose two generating operations of G_{p^2} so that J_B transforms the one into itself and the other into its inverse operation. A change in the generating operations of G_{p^2} does not affect J_A, because this isomorphism transforms every operation in G_{p^2} into its inverse. In this case a single type is thus obtained which is isomorphic with one of those immediately preceding.

In those cases of the continuation when G_{16}/H is isomorphic with a non-cyclic G_4 the method of procedure when setting up the types to be investigated is the same as in the foregoing example. It is only necessary to find out whether

the three types are distinct, which depends on the possibility of changing the generating operations of G_{16} and G_{p^2}.

$H = \{A^2B\}$

G^2_{16}/H is isomorphic with a cyclic G_4. In $G_{8p^2} = \{A, P_1, P_2\}$ J_A must be an isomorphism of order 4 of G_{p^2}. It has been shown previously (P. 9) that for $p \equiv 1 \pmod 4$ we can suppose $J_A = (P_1, P_2; P_1^a, P_2^{a^r})$. a belongs to the exponent 4 (mod p). On the other hand if $p \equiv 3 \pmod 4$ then $J_A = (P_1, P_2; P_2, P_1^{-1})$. J_B is an isomorphism of order 2 necessarily identical with J_{A^2}; thus five distinct types are obtained.

When G_{16}/H is isomorphic with a cyclic G_4 we obtain five types which are always distinct.

$H = \{A^4, B\}$

G^2_{16}/H is isomorphic with a cyclic G_4. J_A can thus be chosen as in the foregoing case, which gives five new distinct types.

$H = \{A^4\}$

G^2_{16}/H is isomorphic with an Abelian G_8 of type (2, 1). In $G_{8p^2} = (A, P_1, P_2)$ the generating operations of G_{p^2} may be chosen so that $J_A = (P_1, P_2; P_1^a, P_2^{a^r})$ or for $p \equiv 3 \pmod 4$. $J_A = (P_1, P_2; P_2, P_1^{-1})$. a belongs to the exponent 4 (mod p). J_B, which is an isomorphism of order 2, must be permutable with J_A. The exponents of J_B satisfy (2).

$J_A = (P_1, P_2; P_1^a, P_2^{a^r})$ is permutable with J_B

if $$ba(a^{r-1} - 1) \equiv ca(a^{r-1} - 1) \equiv \bmod\ p)$$

When $r \neq 1$ is $$b \equiv c \equiv 0$$

$$\because\ a^2 \equiv d^2 \equiv 1.$$

Only three types are obtained for which

$$J_A = (P_1, P_2; P_1^a, P_2) \qquad J_B \equiv (P_1, P_2; P_1^{\pm 1}, P_2^{-1})$$

or $$J_A = (P_1, P_2; P_1^a, P_2^{a^3}) \qquad J_B \equiv (P_1, P_2; P_1^{-1}, P_2)$$

When $r = 1$, J_A is an operation self-conjugate in the whole group of isomorphisms of G_{p^2}. For all J_B we can choose new

generating operations of G_{p^2}, so that B transforms one operation into itself and the other into its inverse operation. J_A is unchanged by this. Thus only one distinct type is obtained, which in this case is isomorphic with the preceding one.

If $p \equiv 3 \pmod 4$ and $J_A = (P_1,\ P_2;\ P_2,\ P_1^{-1})$, it is proved that, except J_{A^2} and the identical isomorphism, there is no isomorphism permutable with J_A whose exponents satisfy the congruences (2). Of the seven groups obtained there are consequently in this case only three distinct ones.

When G_{16}/H is isomorphic with an Abelian G_8 of type (2, 1), the method of investigation is similar to that in the foregoing example. Seven groups are always obtained and it thus remains to be discovered whether they are distinct or not, which depends on the possibility of changing the generating operations of G_{16} and G_{p^2}.

$H = \{B\}$

G^2_{16}/H is isomorphic with a cyclic G_8. In $G_{8p^2} = (A, P_1, P_2)$ J_A must be an isomorphism of order 8. It has been previously shown (P. 9, 10) that we can suppose

$J_A = (P_1, P_2; P_1, P_2^{a^r})$ $\qquad$ a belongs to exp. 8 (mod $p = 8k+1$)

$J_A = (P_1, P_2; P_2, P_1^{\gamma})$ $\qquad$ γ belongs to exp. 4 (mod $p = 8k+5$)

$J_A = (P_1, P_2; P_2, P_1P_2^{\delta})$ $\qquad$ $\delta^2 \equiv -2$ (mod $p = 8k+3$) or

$J_A = (P_1, P_2; P_2, P_1^{-1}P_2^{\delta})$ $\qquad$ $\delta^2 \equiv 2$ (mod $p = 8k+7$)

J_B is the identical isomorphism and 11 types are obtained.

$H = E$

In $G_{8p^2} = \{A,\ P_1,\ P_2\}$ J_A can be chosen as any one of the 11 preceding isomorphisms of order 8 to G_{p^2}. J_B, which is a general isomorphism of order 2, must be permutable with J_A. The exponents of J_B satisfy (2).

$J_A = P_1,\ P_2;\ P_1^{a},\ P_2^{a^r})$ is permutable vith J_B

if $\qquad ba(a^{r-1}-1) \equiv ca(a^{r-1}-1) \equiv 0 \pmod p$

When $r \neq 1$ is $\qquad b \equiv c \equiv 0$

$$\because\ a^2 \equiv d^2 \equiv 1$$

Only 4 distinct types are obtained for which

$$J_A = (P_1,\ P_2;\ P_1^{\alpha},\ P_2) \qquad J_B = (P_1,\ P_2;\ P_1,\ P_2^{-1})$$
$$J_A = (P_1,\ P_2;\ P_1^{\alpha},\ P_2^{\alpha^2}) \qquad J_B = (P_1,\ P_2;\ P_1,\ P_2^{-1})$$
$$J_A = (P_1,\ P_2;\ P_1^{\alpha},\ P_2^{\alpha^3}) \qquad J_B = (P_1,\ P_2;\ P_1^{-1},\ P_2)$$
or $$J_A = (P_1,\ P_2;\ P_1^{\alpha},\ P_2^{\alpha^5}) \qquad J_B = (P_1,\ P_2;\ P_1^{-1},\ P_2)$$

$J_A = (P_1,\ P_2;\ P_1^{\alpha},\ P_2^{\alpha})$ is a self-conjugate operation in the group of isomorphisms of G_{p^2}. The solutions of the congruences (2) give only one type isomorphic with the preceding one.

$J_A = (P_1,\ P_2;\ P_2,\ P_1^{\gamma})$ is permutable with J_B

if $$a \equiv d \quad c \equiv b\gamma \pmod{p}$$

The congruences (2) give

(*i*) $a \equiv 0$ $\qquad \because\ bc \equiv 1$

$\because\ b^2 \equiv \gamma^3$, which has no solutions for $p \equiv 5 \pmod{8}$

(*ii*) $b \equiv 0$ $\qquad \because\ a^2 \equiv 1$ which does not give any group belonging here.

$J_A = (P_1,\ P_2;\ P_2,\ P_1P_2^{\delta})$ is permutable with J_B

if $$b \equiv c \qquad d \equiv a + b\delta \pmod{p}$$

The congruences (2) give

(*i*) $a \equiv d$ $\qquad \because\ 2ab \equiv 0$

When $a \not\equiv 0$ then is $b \equiv 0$

$\because\ a \equiv d \equiv \pm 1$, which gives an isomorphism identical with J_{A^4}

(*ii*) $a \equiv -d$ $\qquad \because\ -2a \equiv b\delta$

$$4a^2 \equiv b^2\delta^2 \qquad \text{but since } \delta^2 \equiv -2$$

we get $$a^2 \equiv -1$$

This congruence is not soluble for $p \equiv 3 \pmod{8}$

$J_A = (P_1,\ P_2;\ P_2,\ P_1^{-1}P_2^{\delta})$ is permutable with J_B

if $$b + c \equiv 0 \qquad d \equiv a + b\delta \pmod{p}.$$

The congruences (2) give

(*i*) $a \equiv d$ $\quad\therefore\ 2ab \equiv 0$

When $a \not\equiv 0$ then is $b \equiv 0$

$\therefore\ a \equiv d \equiv \pm 1$, which gives an isomorphism identical with J_{A^4}

(*ii*) $a \equiv -d$ $\quad\therefore\ -2a \equiv b\delta$

$4a^2 \equiv b^2\delta^2$ but since $\delta^2 \equiv 2$

we get $\quad a^2 \equiv -1$

This congruence has no solution for $p \equiv 7 \pmod 8$

$$G_{16p^2}/G_{p^2} = \{A^4 = B^4 = 1 \quad AB = BA\}$$

$H = \{A, B^2\}$. Two types.
$H = \{A\}$. Five types.
$H = \{A^2, B^2\}$

G^3_{16}/H is isomorphic with a non-cyclic G_4, which here gives only one type for which

$$J_A = (P_1,\ P_2;\ P_1,\ P_2^{-1}) \qquad J_B = (P_1,\ P_2;\ P_1^{-1},\ P_2^{-1})$$

$H = \{B^2\}$

G^3_{16}/H is isomorphic with an Abelian G_8 of type (2, 1). Of the seven types only two are here distinct, viz.

$$J_A = (P_1,\ P_2;\ P_1^a,\ P_2) \qquad J_B = (P_1,\ P_2;\ P_1,\ P_2^{-1})$$

or $$J_A = (P_1,\ P_2;\ P_1^a,\ P_2^{a^3}) \qquad J_B = (P_1,\ P_2;\ P_1,\ P_2^{-1})$$

$H = E$

$G_{8p^2} = \{A, B^2, P_1, P_2\}$ is a self-conjugate sub-group of G_{16p^2}. The isomorphisms J_A and J_{B^2} are permutable and of order 4 and 2 respectively. We can suppose

$$J_A = (P_1,\ P_2;\ P_1^a,\ P_2^{\pm 1}) \qquad J_{B^2} = (P_1,\ P_2;\ P_1^{\pm 1},\ P_2^{-1})$$

$$J_A = (P_1,\ P_2;\ P_1^a,\ P_2^{a^3}) \qquad J_{B^2} = (P_1,\ P_2;\ P_1^{\pm 1},\ P_2^{\mp 1})$$

or $$J_A = (P_1,\ P_2;\ P_1^a,\ P_2^a) \qquad J_{B^2} = (P_1,\ P_2;\ P_1,\ P_2^{-1})$$

J_B is a general isomorphism of order 4 which is permutable with J_A, and J_B^2 is in every particular case identical with J_{B^2}

$J_A = (P_1, P_2; P_1^a, P_2^{a^r}$ is permutable with J_B

if $$ba(a^{r-1}-1) \equiv ca(a^{r-1}-1) \equiv 0 \pmod{p}$$

(*i*) $r \equiv o$ $\quad\because b \equiv c \equiv 0$

Hence it follows that $a^2 \equiv \pm 1 \quad d^2 \equiv -1$

These 8 groups are all isomorphic and only one type is obtained for which

$$J_A = (P_1, P_2; P_1^a, P_2) \qquad J_B = (P_1, P_2; P_1, P_2^a)$$

(*ii*) $r \equiv 1$

J_A is a self-conjugate operation in the group of isomorphisms of G_{p^2}. The exponents of J_B satisfy

$$\left.\begin{array}{ll} a^2+bc \equiv 1 & b(a+d) \equiv 0 \\ d^2+bc \equiv -1 & c(a+d) \equiv 0 \end{array}\right\} \pmod{p}$$

$$\because a+d \not\equiv 0 \qquad b \equiv c \equiv o$$

$$\because a^2 \equiv 1 \qquad d^2 \equiv -1$$

The 4 groups are isomorphic with the preceding ones

(*iii*) $r \equiv 2$ $\quad\because b \equiv c \equiv 0$

Hence follows $\quad a^2 \equiv \pm 1 \quad d^2 \equiv -1$

The groups isomorphic with the preceding ones

(*iv*) $r \equiv 0$ $\quad\because b \equiv c \equiv 0$

Hence it follows that

$$a^2 \equiv 1 \quad d^2 \equiv -1 \quad \text{or} \quad a^2 \equiv -1 \quad d^2 \equiv 1$$

The groups isomorphic with the preceding ones

$$G_{16p^2}/G_{p^2} = \left\{\begin{array}{ll} A^4 = B^2 = C^2 = 1 & AB = BA \\ AC = CA & BC = CB \end{array}\right\}$$

$H = \{A^2, B, C\}$ or $\{A, B\}$. Four types.

$H = \{A\}$ or $\{A^2, B\}$

G^4_{16}/H is isomorphic with a non-cyclic G_4. We obtain three types for which

$$J_B = (P_1,\ P_2;\ P_1;\ P_2^{-1}) \qquad J_C = (P_1,\ P_2;\ P_1^{-1},\ P_2^{-1})$$
$$J_A = (P_1,\ P_2;\ P_1,\ P_2^{-1}) \qquad J_C = (P_1,\ P_2;\ P_1^{-1},\ P_2^{\pm 1})$$

$H = \{B,\ C\}$. Five types.

$H = \{A\}$.

G^4_{16}/H is isomorphic with an Abelian G_8 of type (1, 1, 1). In $G_{4p^2} = \{B,\ C,\ P_1,\ P_2\}$, which is a self-conjugate sub-group of G_{16p^2}, we can suppose

$$J_B = (P_1,\ P_2;\ P_1,\ P_2^{-1}) \qquad J_C = P_1,\ P_2;\ P_1^{-1},\ P_2^{-1}).$$

J_A is always permutable wit J_C and also with J_B if

$$b \equiv c \equiv 0$$

Hence it follows that $\quad a^2 \equiv d^2 \equiv 1$

J_A is thus an isomorphism in the group $\{J_B, J_C\}$. The group of isomorphisms of G_{p^2} thus contains no sub-group isomorphic with an Abelian G_8 of type (1, 1, 1).

$H = \{B\}$

G^4_{16}/H is isomorphic with an Abelian G_8 of type (2, 1). Two types are obtained for which

$$J_A = (P_1,\ P_2;\ P_1^{a},\ P_2) \qquad J_C = (P_1,\ P_2;\ P_1,\ P_2^{-1})$$

or
$$J_A = (P_1,\ P_2;\ P_1^{a},\ P_2^{a^3}) \qquad J_C = (P_1,\ P_2;\ P_1,\ P_2^{-1})$$

$H = E$

Since G^4_{16} contains an Abelian sub-group G_8 of type (1, 1, 1) and since no operation in G_{16} is permutable with every operation in G_{p^2}, the group of isomorphisms of G_{p^2} should contain a sub-group isomorphic with a similar Abelian G_8. It has been previously shown that this is not the case. Consequently there is no such group belonging here.

$$G_{16p^2}/G_{p^2} = \begin{cases} A^2 = B^2 = C^2 = D^2 = 1 & AB = BA \quad AC = CA \\ AD = DA \quad BC = CB & BD = DB \quad CD = DC \end{cases}$$

$H = \{A,\ B,\ C\}$. Two types.

$H = \{A,\ B\}$

G^5_{16}/H is isomorphic with a non-cyclic G_4 and we get a type for which

$$J_C = (P_1, P_2; P_1, P_2^{-1}) \qquad J_D = (P_1, P_2; P_1^{-1}, P_2^{-1})$$

$H = \{A\}$

G^5_{16}/H is isomorphic with an Abelian G_8 of type (111). Consequently there is no group belonging here.

$H = E$

Gives no group because G^5_{16} contains an Abelian G_8 of type (1, 1, 1).

$$G_{16p^2}/G_{p^2} = \{A^8 = B^2 = 1 \qquad B^{-1}AB = A^5\}$$

$H = \{A\}$ or $\{A^2, B\}$. Four types.
$H = \{A^4, B\}$ or $\{BA^2\}$. Ten types.
$H = \{A^2\}$

G^6_{16}/H is isomorphic with a non-cyclic G_4, which gives two types for which

$$J_A = (P_1, P_2; P_1, P_2^{-1}) \qquad J_B = (P_1, P_2; P_1^{-1}, P_2^{\pm 1})$$

$H = \{A^4\}$

G^6_{16}/H is isomorphic with an Abelian G_8 of type (2, 1). Of the seven groups only three are distinct, viz.

$$J_A = (P_1, P_2; P_1^a, P_2) \qquad J_B = (P_1, P_2; P_1^{\pm 1}, P_2^{-1})$$

or $$J_A = (P_1, P_2; P_1^a, P_2^{a^3}) \qquad J_B = (P_1, P_2; P_1, P_2^{-1})$$

$H = E$

In $G_{8p^2} = \{A, P_1, P_2\}$, which is a self-conjugate sub-group of G_{16p^2}, J_A is an isomorphism of order 8. There are (P. 9, 10) 11 of these which cannot be transformed into each other by varying the generating operations of $\{A\}$ and G_{p^2}. J_B is an isomorphism of order 2. The exponents satisfy the congruence (2). The isomorphisms J_A and J_B must also satisfy

$$J_A J_B = J_B J_A^5 \qquad \qquad (3)$$

$J_A = (P_1, P_2; P_1^a, P_2^{a^r})$ and J_B satisfy (3)

if
$$\left.\begin{aligned} a(a^5 - a) \equiv b(a^{5r} - a) \equiv 0 \\ c(a^5 - a^r) \equiv d(a^{5r} - a^r) \equiv 0 \end{aligned}\right\} \pmod p$$

(*i*) $r \not\equiv 5 \pmod 8$ $\qquad \because b \equiv c \equiv 0$

This does not give any isomorphism of G_{p^2}, since $a \equiv 0$ for every value of r.

(*ii*) $r \equiv 5$ $\qquad \because d \equiv 0$

From (2) we get $\qquad bc \equiv 1 \pmod p$

The solutions give isomorphic groups. In order to show this the generating operations of G_{p^2} are varied. The sub-groups $\{P_1\}$ and $\{P_2\}$ are the only ones which are self-conjugate when transformed by A

$$\because O_1 = P_1^x \qquad O_2 = P_2^y$$

Since
$$J_B = (O_1,\ O_2;\ O_2^{b_2},\ O_1^{c_2})$$

we get
$$\left.\begin{aligned} b_1x &\equiv b_2y \\ c_2x &\equiv c_1y \end{aligned}\right\} \pmod p$$

$$\because x \equiv b_2c_1y.$$

Thus we can always find x and y, so that the two types become isomorphic and consequently all the groups are isomorphic with $\{A,\ B,\ P_1,\ P_2\}$ for which

$$J_A = (P_1,\ P_2;\ P_1^a,\ P_2^{a^5}) \qquad J_B = (P_1,\ P_2;\ P_2,\ P_1)$$

$J_A = (P_1,\ P_2;\ P_2,\ P_1^r)$ and J_B satisfy (3)

if
$$c + b\gamma \equiv a + d \equiv 0 \pmod p$$

The congruences (2) give

$$a^2 - b^2\gamma \equiv 1 \pmod p$$

The solutions of this congruence give isomorphic groups. It is only by varying the generating operations for G_{p^2} that we can transform one group, e. g., the one corresponding to $a \equiv 1$ $b \equiv 0$ into any other particular group. We choose, e. g.,

$$O_1 = P_1^{x_1}P_2^{y_1}, \qquad O_2 = P_1^{x_2}P_2^{y_2},$$

which operations generate G_{p^2}, provided that $x_1y_2 - x_2y_1 \not\equiv 0 \pmod p$.

In order that $J_A = (O_1, O_2; O_2, O_1^{\gamma})$, it is necessary that

$$x_1 - y_2 \equiv x_2 - \gamma y_1 \equiv 0 \pmod{p}$$

$J_B = (O_1, O_2; O_1^{a_1}O_2^{b_1}, O_1^{c_1}O_2^{-a_1})$ is the corresponding isomorphism of $a_1^2 - b_1^2\gamma \equiv 1$

$$\because \left.\begin{aligned} x_1(a_1 - 1) + \gamma b_1 y_1 \equiv 0 \\ x_1 b + (a_1 + 1)\, y_1 \equiv 0 \end{aligned}\right\} \pmod{p}$$

These two congruences always have solutions for which moreover

$$x_1^2 - \gamma y_1^2 \not\equiv 0 \pmod{p}$$

We obtain a single type for which

$$J_A = (P_1, P_2; P_2, P_1^{\gamma}) \qquad J_B = (P_1, P_2; P_1, P_2^{-1})$$

$J_A = (P_1, P_2; P_2, P_1P_2^{\delta})$ and J_B satisfy (3)

if
$$\left.\begin{aligned} b + c \equiv a + b\delta + d \equiv 0 \\ a + d + c\delta \equiv b + 2d\delta + c \equiv 0 \end{aligned}\right\} \pmod{p}$$

$$\because\ d \equiv 0 \qquad \delta(b - c) \equiv 0$$

Hence it follows that $\qquad b \equiv c \equiv 0$

Consequently no such isomorphism exists[1]

$J_A = (P_1, P_2; P_2, P_1^{-1}P_2^{\delta})$ and J_B satisfy (3)

if
$$\left.\begin{aligned} b - c \equiv a + d + b\delta \equiv 0 \\ a + d - c\delta \equiv b - c + 2d\delta \equiv 0 \end{aligned}\right\} \pmod{p}$$

$$\because\ d \equiv 0 \qquad b + c \equiv 0$$

Hence it follows that $\qquad b \equiv c \equiv 0$

Consequently no such isomorphism exists[1]

$$G_{16p^2}/G_{p^2} = \left\{\begin{matrix} A^2 = B^2 = C^4 = 1 & B^{-1}AB = AC^2 \\ AC = CA & BC = CB \end{matrix}\right\}$$

$H = \{C, A\}$, $\{AC, B\}$ or $\{AC, BC\}$. Six types.

$H = \{C\}$, $\{AC\}$ or $\{A, C^2\}$

[1] This is according to the footnote (Page 10).

G^7_{16}/H is isomorphic with a non-cyclic G_4. We obtain six types for which

$$J_A = (P_1, P_2; P_1, P_2^{-1}) \qquad J^B = (P_1, P_2; P_1^{-1}, P_2^{-1})$$
$$J_A = J_C = (P_1, P_2; P_1, P_2^{-1}) \qquad J_B = (P_1, P_2; P_1^{-1}, P_2^{\pm 1})$$
$$J_A = J_C = (P_1, P_2; P_1^{-1}, P_2^{-1}) \quad J_B = (P_1, P_2; P_1^{-1}, P_2)$$
$$\text{or} \qquad J_B = (P_1, P_2; P_1^{-1}, P_2) \qquad J_C = (P_1, P_2; P_1^{\pm 1}, P_2^{-1})$$

$H = \{C^2\}$

G^7_{16}/H is isomorphic with an Abelian G_8 of type (111). It has been previously shown that in such cases there is no group

$H = E$

$G_{8p^2} = \{C, B, P_1, P_2\}$ is a self-conjugate sub-group of G_{16p^2} Since $\{C, B\}$ is an Abelian G_8 of type (2, 1), we obtain, as before (P. 14), the 7 possible types G_{8p^2}. If A and C are retained as generating operations and B is exchanged for BC^2, it is plain that three of these can be excluded.

$$\because G_{8p^2} = \{C, B, P_1, P_2\}, \text{ for which}$$

$$J_C = (P_1, P_2; P_1^a, P_2^{a^r}) \quad J_B = (P_1, P_2; P_1, P_2^{-1})$$

J_A is an isomorphism of order 2. The exponents thus satisfy the congruences (2).

Necessary conditions:

$$J_B J_A = J_A J_B J_{C^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (4)$$
$$J_A J_C = J_C J_A$$

The relation (4) is satisfied provided that

$$\left.\begin{aligned} a(a^2 - 1) \equiv c(a^2 + 1) \equiv 0 \\ b(a^{2r} + 1) \equiv d(a^{2r} - 1) \equiv 0 \end{aligned}\right\} \pmod{p}$$

(*i*) $\quad r \equiv 0, 2 \pmod 4$

$\because a \equiv b \equiv 0$, which does not give any isomorphism of $G^1{}_{p^2}$

[1] It is also easy to prove this directly. The central of G^7_{16} is $\{C\}$. Neither J_B nor J_{BC^2} can thus be an isomorphism self-conjugate in the group of isomorphisms of G_{p^2}.

(*ii*) $r \equiv 1,\ 3$ $\qquad\qquad \because\ a \equiv d \equiv 0$

The congruences (2) give

$$bc \equiv 1 \pmod p$$

The isomorphisms J_C and J_A are permutable only for $r \equiv 1$. The solutions give isomorphic types. The only sub-groups in G_{p^2} which are self-conjugate when transformed by B are $\{P_1\}$ and $\{P_2\}$. We must therefore choose

$$O_1 = P_1{}^x,\ O_2 = P_2{}^y$$

This choice of new operations does not alter the type of J_C. The exponents x and y can afterwards be determined in the manner required. We get a single type for which

$J_A = (P_1,\ P_2;\ P_2,\ P_1)\quad J_B = (P_1,\ P_2;\ P_1,\ P_2{}^{-1})\quad J_C = (P_1,\ P_2;\ P_1{}^a,\ P_2{}^a)$

$$G_{16p^2}G/_{p^2} = \{A^4 = B^4 = 1 \qquad B^{-1}AB = A^3\}$$

$H = \{A,\ B^2\}$ or $\{B,\ A^2\}$. Four types.

$H = \{A\}$. Five types.

$H = \{A^2,\ B^2\}$

G^8_{16}/H is isomorphic with a non-cyclic G_4, which gives two types for which

$$J_A = (P_1,\ P_2;\ P_1{}^{-1},\ P_2{}^{\pm 1}) \qquad J_B = (P_1,\ P_2;\ P_1,\ P_2{}^{-1})$$

$H = \{A^2\}$

G^8_{16}/H is isomorphic with an Abelian G_8 of type (2, 1). Of the seven groups only two are distinct, viz.

$$J_B = (P_1,\ P_2;\ P_1{}^a,\ P_2{}^{a^{4,3}}) \qquad J_A = (P_1,\ P_2;\ P_1,\ P_2{}^{-1})$$

$H = \{B^2\}$

G^8_{16}/H is isomorphic with a dihedral group. In $G_{4p^2} = \{A,\ P_1,\ P_2\}$, which is a self-conjugate sub-group of G_{16p^2}, we can suppose

$$J_A = (P_1,\ P_2;\ P_1^{a},\ P_2^{a^r}) \qquad a^4 \equiv 1 \pmod p$$

or

$$J_A = (P_1,\ P_2;\ P_2,\ P_1^{-1}) \qquad p \equiv 3 \pmod 4)$$

J_B is an isomorphism of order 2 which besides the congruences (2) satisfies

$$J_A J_B = J_B J_{A^3} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

$J_A = (P_1,\ P_2;\ P_1^{a},\ P_2^{a^r})$ and J_B satisfy (5)

if

$$\left.\begin{aligned} a(a^3 - a) \equiv c(a^3 - a^r) \equiv 0 \\ b(a^{3r} - a) \equiv d(a^{3r} - a^r) \equiv 0 \end{aligned}\right\} \pmod p$$

(*i*) $r \equiv 0$, 1 or 2 (mod 4)

$\because\ a \equiv b \equiv c \equiv 0$ which does not give any isomorphism of G_{p^2}

(*ii*) $r \equiv 3$ $\qquad \because\ a \equiv d \equiv 0$

The congruences (2) give

$$bc \equiv 1 \pmod p)$$

The solutions of this congruence give isomorphic types. We get $\{A,\ B,\ P_1,\ P_2\}$ for which

$$J_A = (P_1,\ P_2;\ P_1^{a},\ P_2^{a^3}) \qquad J_B = (P_1,\ P_2;\ P_2,\ P_1)$$

$J_A = (P_1,\ P_2;\ P_2,\ P_1^{-1})$ and J_B satisfy (5)

if

$$b - c \equiv 0 \qquad a + d \equiv 0 \pmod p)$$

The congruences (2) give

$$a^2 + b^2 \equiv 1$$

The solutions of this congruence give isomorphic types and we obtain a single type for which

$$J_A = (P_1,\ P_2;\ P_2,\ P_1^{-1}) \qquad J_B = (P_1,\ P_2;\ P_1,\ P_2^{-1})$$[1]

When G_{16}/H is isomorphic with a dihedral group the procedure in setting up the groups to be investigated is the same as in the foregoing example. The type of G_{16} does

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[1] These defining relations of course give the immediately preceding type if we suppose $p \equiv 1$ (mod 4).

not affect the number of groups, since we need only alter the generating operations of G_{p^2}.

$H = \{A^2 B^2\}$

G^8_{16}/H is isomorphic with the quaternion-group. In $G_{4p^2} = \{A,\ P_1,\ P_2\}$ we can suppose $J_A = (P_1,\ P_2;\ P_1^{\alpha},\ P_2^{\alpha^r})$ or $(P_1,\ P_2;\ P_2,\ P_1^{-1})$.

J_B is an isomorphism of order 4 which satisfies

$$J_A J_B = J_B J_{A^3} \quad \dots\dots\dots \quad (6)$$

$$J_{A^2} = J_{B^2} \quad \dots\dots\dots \quad (7)$$

$J_A = (P_1,\ P_2;\ P_1^{\alpha},\ P_2^{\alpha^r})$ satisfies (6)

if $\qquad r \equiv 0 \pmod 4 \qquad a \equiv d \equiv 0 \pmod p$

Relation (7) gives $\qquad bc \equiv -1 \pmod p$

We obtain a single type for which

$$J_A = (P_1,\ P_2;\ P_1^{\alpha},\ P_2^{\alpha^3}) \qquad J_B = (P_1,\ P_2;\ P_2,\ P_1^{-1})$$

$J_A = (P_1,\ P_2;\ P_2,\ P_1^{-1})$ and J_B satisfy (6)

if $\qquad b - c \equiv 0 \quad a + d \equiv 0 \pmod p$

Relation (7) gives $\qquad a^2 + b^2 \equiv -1 \pmod p$

We obtain a single type for which

$$J_A = (P_1,\ P_2;\ P_2,\ P_1^{-1}) \qquad J_B = (P_1,\ P_2;\ P_1^a P_2^b,\ P_1^b P_2^{-a})$$[1]

When G_{16}/H is isomorphic with the quaternion-group the method of procedure when setting up the groups to be investigated is the same as in the foregoing example. The type of G_{16} does not affect the number of groups, as it is only necessary to vary the generating operations of G_{p^2}.

$H = E$

$G_{8p^2} = \{A,\ B^2,\ P_1,\ P_2\}$ is a self-conjugate sub-group of G_{16p^2} Since $\{A,\ B^2\}$ is an Abelian G_8 of type (2, 1), we get seven G_{8p^2} (P. 15). If B is retained, G^8_{16} can be generated by

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[1] These defining relations of course give the immediately preceding type if we suppose $p \equiv 1 \pmod 4$.

$A_1 = A,\ A^3,\ AB^2$ or A^3B^2, which shows that three of these can be excluded

$$\because\ G_{8p^2} = \{A,\ B^2,\ P_1,\ P_2\},\ \text{for which}$$

$$J_A = (P_1,\ P_2;\ P_1^{a},\ P_2) \qquad J_{B^2} = (P_1,\ P_2;\ P_1^{\pm 1},\ P_2^{-1})$$

or $J_A = (P_1,\ P_2;\ P_1^{a},\ P_2^{a^3}) \qquad J_{B^2} = (P_1,\ P_2;\ P_1^{\pm 1},\ P_2^{\mp 1})$

J_B is an isomorphism of order 4 which satisfies

$$J_A J_B = J_B J_{A^3}$$
$$J_B^2 = J_{B^2}$$

(*i*) $r \equiv 0 \pmod 4$ $\quad \because\ a \equiv b \equiv c \equiv 0 \pmod p$, which does not give any isomorphism of G_{p^2}

(*ii*) $r \equiv 3$ $\qquad \because\ a \equiv d \equiv 0$

The exponents b and c of J_B must then satisfy at the same time $bc \equiv \pm 1$, which is impossible when p is odd.

$$G_{16p^2}/G_{p^2} = \begin{Bmatrix} A^4 = B^2 = C^2 = 1 & B^{-1}AB = A^3 \\ BC = CB & AC = CA \end{Bmatrix}$$

$H = \{A^2,\ B,\ C\},\ \{A,\ C\}$ or $\{A,\ B\}$. Six types.

$H = \{A\},\ \{A^2,\ B\}$ or $\{A^2,\ C\}$

G_{16}^{9}/H is isomorphic with a non-cyclic G_4. We obtain six types for which

$$J_C = (P_1,\ P_2;\ P_1^{\pm 1},\ P_2^{-1}) \quad J_B = (P_1,\ P_2;\ P_1^{-1},\ P_2)$$
$$J_A = (P_1,\ P_2;\ P_1,\ P_2^{-1}) \quad J_C = (P_1,\ P_2;\ P_1^{-1},\ P_2^{\pm 1})$$
$$J_A = (P_1,\ P_2;\ P_1^{\pm 1},\ P_2^{-1}) \quad J_B = (P_1,\ P_2;\ P_1^{-1},\ P_2)$$

$H = \{A^2\}$

G_{16}^{9}/H is isomorphic with an Abelian G_8 of type (1, 1, 1). Thus there is no group belonging here.

$H = \{C\}$

G_{16}^{9}/H is isomorphie with a dihedral group. As was previously shown, there exists for all p a type for which

$$J_A = (P_1,\ P_2;\ P_2,\ P_1^{-1}) \qquad J_B = (P_1,\ P_2;\ P_1,\ P_2^{-1})$$

$H=E$

This is not possible, since G^{9}_{16} contains an Abelian G_8 of type (1, 1, 1)

$$G_{16p^2}/G_{p^2}=\left\{\begin{matrix} A^4=B^2=C^2=1 & B^{-1}AB=AC \\ AC=CA & BC=CB \end{matrix}\right\}$$

$H=\{A^2,\ B,\ C\}$ or $\{A,\ C\}$. Four types.

$H=\{A^2,\ C\}$

G^{10}_{16}/H is isomorphic with a non-cyclic G_4, which gives two types for which

$$J_A=(P_1,\ P_2;\ P_1,\ P_2^{-1})\qquad J_B=(P_1,\ P_2;\ P_1^{-1},\ P_2^{\pm 1})$$

$H=\{B,\ C\}$. Five types.

$H=\{A^2\}$

G^{10}_{16}/H is isomorphic with an dihedral group. We obtain a single type for which

$$J_A=(P_1,\ P_2;\ P_2^{-1},\ P_1^{-1})\qquad J_B=(P_1,\ P_2;\ P_1,\ P_2^{-1})$$
$$J_C=(P_1,\ P_2;\ P_1^{-1},\ P_2^{-1})$$

$H=\{C\}$

G^{10}_{16}/H is isomorphic with an Abelian G_8 of type (2, 1). We obtain two types for which

$$J_A=(P_1,\ P_2;\ P_1^{a},\ P_2^{a^4,3})\qquad J_B=(P_1,\ P_2;\ P_1,\ P_2^{-1})$$

$H=E$

This is impossible, since G^{10}_{16} contains an Abelian G_8 of type (1, 1, 1).

$$G_{16p^2}/G_{p^2}=\left\{\begin{matrix} A^4=B^4=C^2=1 & A^2=B^2 & B^{-1}AB=A^3 \\ & AC=CA & BC=CB \end{matrix}\right\}$$

$H=\{A,\ C\}$ or $\{A,\ B\}$. Four types.

$H=\{A\}$ or $\{A^2,\ C\}$

G^{11}_{16}/H is isomorphic with a non-cyclic G_4. We obtain three types for which

$$J_B = (P_1,\ P_2;\ P_1,\ P_2^{-1}) \qquad J_C = (P_1,\ P_2;\ P_1^{-1},\ P_2^{\pm 1})$$
$$J_A = (P_1,\ P_2;\ P_1,\ P_2^{-1}) \qquad J_B = (P_1,\ P_2;\ P_1^{-1},\ P_2)$$

$H = \{A^2\}$

G^{11}_{16}/H is isomorphic with an Abelian G_8 of type (1, 1, 1). Thus there is no group belonging here.

$H = \{C\}$

G^{11}_{16}/H is isomorphic with the quaternion-group. We obtain a single type for which

$$J_A = (P_1,\ P_2;\ P_2,\ P_1^{-1}) \qquad J_B = (P_1,\ P_2;\ P_1^a P_2^b,\ P_1^b P_2^{-a})$$

$H = E$

In $G_{8p^2} = \{A,\ B,\ P_1,\ P_2\}$, which is a self-conjugate sub-group of G_{16p^2}, we can choose J_A and J_B for every value of p as in the preceding type. The exponents a and b are a fixed solution of $a^2 + b^2 \equiv -1 \pmod p$

$J_C = (P_1,\ P_2;\ P_1^{x_1} P_2^{y_1},\ P_1^{x_2} P_2^{y_2})$ is an isomorphism of order 2 which is permutable with J_A

if
$$y_1 + x_2 \equiv x_1 - y_2 \equiv 0 \pmod p$$
$$\because\ 2x_1 y_1 \equiv 0 \quad x_1^2 - y_1^2 \equiv 1$$

(*i*) $x_1 \equiv 0$ $\qquad \because\ y_1^2 \equiv -1$

For $p \equiv 1 \pmod 4$ this gives an isomorphism which is not permutable with J_B

(*ii*) $y_1 \equiv 0$ $\qquad \because\ x_1^2 \equiv 1$, which does not give any group belonging here

$$G_{16p^2}/G_{p^2} = \left\{A^8 = B^2 = 1 \quad B^{-1}AB = A^{-1}\right\}$$

$H = \{A\}$ or $\{A^2,\ B\}$. Four types.

$H = \{A^2\}$

G^{12}_{16}/H is isomorphic with a non-cyclic G_4. We get two types for which

$$J_A = (P_1,\ P_2;\ P_1^{\pm 1},\ P_2^{-1}) \qquad J_B = (P_1,\ P_2;\ P_1^{-1},\ P_2)$$

$H = \{A^4\}$

G_{16}^{12}/H is isomorphic with a dihedral group. We get a single type for which

$$J_A = (P_1, P_2; P_2, P_1^{-1}) \qquad J_B = (P_1, P_2; P_1, P_2^{-1})$$

$H = E$

We can suppose for every p-value that in $G_{8p^2} = \{A^2, B, P_1, P_2\}$

$$J_{A^2} = (P_1, P_2; P_2, P_1^{-1}) \text{ and } J_B = (P_1, P_2; P_1, P_2^{-1})$$

J_A is a general isomorphism of order 8 which satisfies

$$J_A J_B = J_B J_{A^7} \quad \dots \dots \dots \quad (8)$$

$$J_A^2 = J_{A^2} \quad \dots \dots \dots \quad (9)$$

J_A satisfies (8) if

$$a + c \equiv b - d \equiv 0 \pmod p$$

From (9) we get $\quad -c \equiv a \equiv b \equiv d$

$\because\ 2c^2 \equiv 1$, which is soluble for $p \equiv \pm 1 \pmod 8$

The two solutions give a distinct type for which

$$J_A = (P_1, P_2; P_1^c P_2^c, P_1^{-c} P_2^c) \qquad J_B = (P_1, P_2; P_1, P_2^{-1})$$

$$G_{16p^2}/G_{p^2} = \left\{A^8 = B^2 = 1 \quad B^{-1}AB = A^3\right\}$$

$H = \{A\}$, $\{A^2, B\}$ or $\{A^2, BA\}$. Six types.

$H = \{A^2\}$

G_{16}^{13}/H is isomorphic with a non-cyclic G_4. We get three types for which

$$J_A = (P_1, P_2; P_1, P_2^{-1}) \qquad J_B = (P_1, P_2; P_1^{-1}, P_2^{\pm 1})$$

or $\quad J_A = (P_1, P_2; P_1^{-1}, P_2^{-1}) \qquad J_B = (P_1, P_2; P_1, P_2^{-1})$

$H = \{A^4\}$

G_{16}^{13}/H is isomorphic with a dihedral group. We get a single type for which

$$J_A = (P_1, P_2; P_2, P_1^{-1}) \qquad J_B = (P_1, P_2; P_1, P_2^{-}$$

$H = E$

We can suppose for every value of p that in $G_{8p^2} = \{A^2, B, P_1, P_2\}$

$$J_{A^2} = (P_1, P_2; P_2, P_1^{-1}) \text{ and } J_B = (P_1, P_2; P_1, P_2^{-1})$$

J_A is a general isomorphism of order 8 which satisfies

$$J_A J_B = J_B J_{A^3} \quad \ldots\ldots\ldots \quad (10)$$

$$J_A^2 = J_{A^2} \quad \ldots\ldots\ldots \quad (11)$$

J_A satisfies (10) if

$$a - c \equiv b + d \equiv 0 \pmod{p}$$

From (11) we get $\quad a \equiv -b \equiv c \equiv d$

$\because\ 2c^2 \equiv -1$, which is soluble for $p \equiv 1, 3 \pmod{8}$

The two solutions give a distinct type for which

$$J_A = (P_1, P_2; P_1^a P_2^{-a}, P_1^a P_2^a) \qquad J_B = (P_1, P_2; P_1, P_2^{-1})$$

$$G_{16p^2}/G_{p^2} = \left\{A^8 = B^4 = 1 \quad A^4 = B^2 \quad B^{-1}AB = A^7\right\}$$

$H = \{A\}$ or $\{A^2, B\}$. Four types.

$H = \{A^2\}$

G^{14}_{16}/H is isomorphic with a non-cyclic G_4. We get two types for which

$$J_A = (P_1, P_2; P_1^{\pm 1}, P_2^{-1}) \qquad J_B = (P_1, P_2; P_1^{-1}, P_2)$$

$H = \{A^4\}$

G^{14}_{16}/H is isomorphic with a dihedral group. We get a single type for which

$$J_A = (P_1, P_2; P_2; P_1^{-1}) \qquad J_B = (P_1, P_2; P_1, P_2^{-1})$$

$H = E$

We can suppose for every value of p that in $G_{8p^2} = \{A^2, B, P_1, P_2\}$

$$J_{A^2} = (P_1, P_2; P_2, P_1^{-1}) \quad J_B = (P_1, P_2; P_1^a P_2^b, P_1^b P_2^{-a})$$

The exponents a and b are a fixed solution of $a^2 + b^2 \equiv -1$

$J_A = (P_1, P_2;\ P_1^{x_1}P_2^{y_1},\ P_1^{x_2}P_2^{y_2})$ is an isomorphism of order 8 which satisfies

$$J_A J_B = J_B J_{A^7}$$
$$J_A^2 = J_{A^2}$$

From these relations we get

$$x_1 \equiv y_1 \equiv -x_2 \equiv y_2 \qquad 2x_1^2 \equiv 1 \pmod{p}$$

We obtain for $p \equiv \pm 1 \pmod{8}$ a single type for which

$J_A = (P_1, P_2;\ P_1^{x_1}, P_2^{x_1}, P_1^{-x_1}P_2^{x_1})$ $J_B = (P_1, P_2;\ P_1^{a}P_2^{b},\ P_1^{b}P_2^{-a})$

IV.

Those G_{16p^2} which have a self-conjugate G_{16} and more than one G_{p^2}.

If the operations in G_{16} are transformed with the operations in G_{p^2} the same operations are obtained in another order. Every operation in G_{p^2} thus corresponds to an isomorphism of G_{16} and the group of isomorphisms of the latter must contain a sub-group of order p at least. G_{p^2} would otherwise be self-conjugate in G_{16p^2}. Groups belonging here can thus exist only for G_{16}^r $(r = 3, 4, 5, 7, 11)$

(*i*) G_{p^2} *non-cyclic*

$r \neq 5$ $\qquad\qquad \because\ p = 3$

Every operation in a sub-group G_p, e. g. $\{P_2\}$, is permutable with every operation in G_{16}. The groups sought are thus obtained as the direct product of the corresponding $G_{16.3}$[1] and a cyclic G_3. We obtain four types with the following defining relations

$\{G_{16}^3, G_9\}$ for which $J_{P_1} = (A\ \ B\ \ A^3B^3)$

$\{G_{16}^4, G_9\}$ for which $J_{P_1} = (A)\ (B\ \ C\ \ BC)$

[1] Levavasseur, Les groups d'ordre $16p$. Toulouse Ann. Vol. 5 (1903).

$\{G^{7}_{16}, G_9\}$ for which $J_{P_1} = (C)\ (A\ B\ ABC)$

$\{G^{11}_{16}, G_9\}$ for which $J_{P_1} = (C)\ (A\ B\ AB)$

$r = 5$ $\because p = 3,\ 5$ or 7

$G_{16 \cdot 25}$ and $G_{16 \cdot 49}$ are obtained as the direct product of the corresponding G_{16p} and a cyclic G_p. Two types are obtained

$\{G^{5}_{16}, G_{25}\}$ for which $J_{P_1} = (A\ B\ C\ D\ ABCD)$

$\{G^{5}_{16}, G_{49}\}$ for which $J_{P_1} = (A)\ (B\ C\ D\ BC\ CD\ BCD\ BD)$

The investigated groups $G_{16 \cdot 9}$ contain 4 or 16 G_9. If P_2 is permutable with every operation in G^{5}_{16} and $G_{16 \cdot 9}$ contains 4 G_9, P_1 must be permutable with every operation in a subgroup G_4 e. g. $\{A, B\}$. The other operations in G^{5}_{16} are permuted cyclically. On the other hand if $G_{16 \cdot 9}$ contains 16 G_9, then P_1 permutes all the operations in G^{5}_{16} in cyclic sequences. The group of isomorphisms of G^{5}_{16} contains only one self-conjugate sequence of non-cyclic G_9. Thus there exists only one group in which both J_{P_1} and J_{P_2} are separated from the identical isomorphism. As generating isomorphisms we can choose those previously obtained of order 3, since these generate a non-cyclic G_9. Three types are obtained

$\{G^{5}_{16}, G_9\}$ for which $J_{P_1} = (A)\ (B)\ (B\ C\ BC)$

$\{G^{5}_{16}, G_9\}$ for which $J_{P_1} = (A\ B\ AB)\ (B\ C\ BC)$

$\{G^{5}_{16}, G_9\}$ for which $J_{P_1} = (A)\ (B)\ (B\ C\ BC)$ and $J_{P_2} = (A\ B\ AB)\ (B\ C\ BC)$.

(*ii*) G_{p^2} *cyclic*

The groups of isomorphisms of G^{r}_{16} ($r = 3,\ 4,\ 5,\ 7,\ 11$) contain no cyclic G_{p^2}. Every operation in $\{P^p\}$ must thus be permutable with every operation in G^{r}_{16}. Thus we get 8 G_{16p^2} direct from the foregoing relations.

V.

The G_{16p^2} which contain no self-conjugate G_{16} or G_{p^2}.

(*i*) $p = 7$

All the groups $G_{16.49}$ contain 8 G_{49}. These 8 G_{49} have a common sub-group G_7, which is self-conjugate in $G_{16.49}$. The factor-group $G_{16.49}/G_7 = \Gamma_{16.7}$ which is formed by this has no self-conjugate sub-group of order 7. $G_{16.49}$ would otherwise have a self-conjugate G_{49}, which is contrary to the hypothesis. $\Gamma_{16.7}$ thus[1] has a self-conjugate G_{16} which is necessarily (Page 33) an Abelian group of type (1, 1, 1, 1). These groups can thus only exist when the conjugated sequence of groups G_{16} are Abelian groups of type (1, 1, 1, 1). Since the factor-group $\Gamma_{16.7}$ has a self-conjugate G_{56}, all $G_{16.49}$ contain a self-conjugate $G_{8.49}$. This $G_{8.49}$ cannot have a self-conjugate G_{49}. G_{49} would, in that case, be self-conjugate in $G_{16.49}$ which is contrary to the hypothesis. It contains, however, a self-conjugate Abelian G_8 of type (1, 1, 1).

$G_{8.49} = \{A, B, C, P\}$ for which $J_P = (A \ B \ C \ AB \ . \ . \ . \ .)$
$G_{8.49} = \{A, B, C, P_1, P_2\}$ for which $J_{P_1} = 1$ and
$J_{P_2} = (A \ B \ C \ AB \ . \ . \ .)$

(*a*) G_{49} *cyclic*

$G_{16} = \{A, B, C, D\}$. D permutes the 8 cyclic sub-groups G_{49}

$$\because \ D^{-1}PD = P^x A^y B^z C^v$$
$$D^{-2}PD^2 = P$$

Hence it follows that $\qquad x^2 \equiv 1 \pmod{49}$

$x \equiv 1 \qquad\qquad \because \ y, z$ and v arbitrary

The groups are all isomorphic and contain a self-conjugate G^5_{16}. In order to shew this it is only necessary to choose as

[1] According to Levavasseur the groups $G_{16.7}$ have either a self-conjugate G_{16} or a self-conjugate G_7.

new generating operations instead of D the 8 operations in G_{16} which were not found in $G_8 = \{A, B, C\}$

$x \equiv -1$
$$D^{-1}PD = P^{-1}U$$
$$D^{-2}PD^2 = UPU$$
$$\therefore U = E$$

When D transforms P in the inverse operation, no isomorphism of order 2 of $G_{8.49}$ is formed. Consequently no new type exists

(*b*) G_{49} *non-cyclic*

Each of the 8 G_{49} can be generated by P_1 and an operation that is the product of P_2 and an operation in $G_8 = \{A, B, C\}$

$$\because D^{-1}P_1D = P_1^{\alpha}$$
$$D^{-1}P_2D = P_1^{\gamma}P_2^{\delta}U$$

From these relations it follows, since D^2 is permutable with P_1 and P_2

$$\alpha^2 \equiv \delta^2 \equiv 1 \qquad \gamma(\alpha + \delta) \equiv 0 \pmod 7 \quad . \quad . \quad . \ (1)$$

$G_8 = \{A, B, C\}$ is self-conjugate in $G_{16.49}$, and the isomorphisms corresponding to D, P_1 and P_2 must satisfy

$$J_{P_1}J_D = J_DJ_{P_1}{}^{\alpha}$$
$$J_{P_2}J_D = J_DJ_{P_1}{}^{\gamma}J_{P_2}{}^{\delta}$$

$J_{P_1} = J_D = J_E$ $\qquad \because \delta \equiv 1$

From (1) we get $\qquad \alpha^2 \equiv 1 \quad \gamma(\alpha + 1) \equiv 0$

$$\because \begin{cases} \alpha \equiv 1 \\ \gamma \equiv 0 \end{cases} \text{or} \begin{cases} \alpha \equiv -1 \\ \gamma \text{ arbitrary} \end{cases}$$

1.
$$D^{-1}P_1D = P_1$$
$$D^{-1}P_2D = P_2U$$

D^2 is permutable with P_2 independent of the value of U. All the groups contain a self-conjugate G^5_{16}

2.
$$D^{-1}P_1D = P_1^{-1}$$
$$D^{-1}P_2D = P_1^{\gamma}P_2U$$

Here we can choose $U = E$ independent of γ. The 7 groups for different values of γ are all isomorphic. We thus obtain a single type of $G_{16.49}$, defined by the relations

$$A^2 = B^2 = C^2 = D^2 = P_1^7 = P_2^7 = 1 \quad AB = BA \quad AC = CA \quad AD = DA$$
$$BC = CB \quad BD = DB \quad CD = DC \quad P_1P_2 = P_2P_1$$
$$AP_1 = P_1A \quad BP_1 = P_1B \quad CP_1 = P_1C \quad D^{-1}P_1D = P_1^{-1}$$
$$P_2^{-1}AP_2 = B \quad P_2^{-1}BP_2 = C \quad P_2^{-1}CP_2 = AB \quad DP_2 = P_2D$$

The other six types are obtained from this if, instead of P_2, we choose as the new generating operation $P_1^tP_2$, where $2t + \gamma \equiv 0 \pmod 7$

(*ii*) $p \equiv 5$

Sylow's theorem shews that all the groups $G_{16.25}$ contain 16 G_{25}. The sub-group in $G_{16.25}$ which contains G_{25} self-conjugate is thus of order 25 and coincides with the group itself. Since G_{25} is Abelian, the conjugated sequence of groups G_{16} can thus contain not more than one group. Consequently no group belonging here exists.

(*iii*) $p = 3$

The groups $G_{16.9}$ contain a self-conjugated sequence with 4 or 16 G_9. In the case of 16 G_9 each of these G_9 is self-conjugate in a sub-group of $G_{16.9}$ which coincides with the group G_9 itself. Since G_9 is Abelian, G_{16} must in this case necessarily be self-conjugate in $G_{16.9}$, which is contrary to the hypothesis. The conjugated sequence of groups G_9 thus contains only 4 G_9 which have a common sub-group G_3. This G_3 is self-conjugate in $G_{16.9}$. The factor-group $G_{16.9}/G_3 = \Gamma_{48}$ which is formed by this contains 3 G_{16} or 1 G_{16}.

If Γ_{48} contains 1 G_{16}, $G_{16.9}$ has a self-conjugate G_{48}. This G_{48} has 3 G_{16} or 1 G_{16}. In the case of 1 G_{16} this is also self-conjugate in $G_{16.9}$, which is contrary to the hypothesis. If, however, G_{48} has 3 G_{16}, these form a conjugated sequence in $G_{16.9}$. These 3 G_{16} have necessarily a common G_8 which is self-conjugate in $G_{16.9}$. Besides the self-conjugate G_3 there

thus exists also a self-conjugate G_8 which leads to a self-conjugate G_{24} in $G_{16.9}$.

If Γ_{48} contains 3 G_{16}, these must have a common G_8 self-conjugate in Γ_{48}, and consequently $G_{16.9}$ has also in this case a self-conjugate G_{24}.

The factor-group $G_{16.9}/G_{24} = \Gamma_6$ has a self-conjugate H_3. $G_{16.9}$ has thus a self-conjugate G_{72} which, moreover, can be chosen in such a way that every operation in G_3 is self-conjugate. G_{72} has 4 G_9 which form a conjugated sequence in $G_{16.9}$.

G_9 *cyclic*

The conjugated sequence of groups G_8 to G_{72} contains 1, 3 or 9 G_8. In the case 1 G_8 there exists groups containing 4 G_9 only when the group of isomorphisms of G_8 is divisible by 3. A single group[1] exists which lacks both self-conjugate G_9 and G_8. In this group P^3 is not self-conjugate and $G_{16.9}$ contains at the same time a self-conjugate G_{72} of another type.
We may thus take

$$G^1_{72} = \begin{Bmatrix} & T_1^2 = T_2^2 = T_3^2 = P^9 = 1 & \\ T_1T_2 = T_2T_1 & T_1T_3 = T_3T_1 & T_2T_3 = T_3T_2 \\ P^{-1}T_1P = T_1 & P^{-1}T_2P = T_3 & P^{-1}T_3P = T_2T_3 \end{Bmatrix}$$

$$G^2_{72} = \begin{Bmatrix} T_1^4 = T_2^4 = P^9 = 1 \quad T_1^2 = T_2^2 \quad T_1^{-1}T_2T_1 = T_2^3 \\ P^{-1}T_1P = T_2 \quad P^{-1}T_2P = T_1T_2 \end{Bmatrix}$$

G^1_{72}.

This group can appear as a self-conjugate sub-group of $G_{16.9}$ only when the conjugated sequence of groups G_{16} to

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[1] As I was not aware that Mr Tripp had set up the groups $G_{p^3q^2}$, I first gave the defining relations for all G_{8p^2}. The $G_{p^3q^2}$ which lack both self-conjugate G_{p^3} and G_{q^2} are all, as can easily be shewn, of order 72. Mr Tripp's investigation is in this case not correct. He sets up the defining relations only for three groups, while in reality there are four such G_{72}.

$G_{16.9}$ is one of the types G^4_{16}, G^5_{16}, G^9_{16} or G^{10}_{16}. $G_8 = \{T_1, T_2, T_3\}$ is a self-conjugate sub-group of $G_{16.9}$.

(*a*) G^4_{16} contains an Abelian G_8 of type (1, 1, 1), viz. $\{A^2, B, C\}$. P is permutable with an operation in G_8. For this we can choose A^2 or B. G_8 has 7 G_4, one of which is permutable with P. This G_4 cannot contain the operation which is permutable with P.

$\because\ J_P = (A^2)\ (B\ C\ BC) \ldots$, $(B)\ (A^2\ C\ A^2C) \ldots$ or $(B)\ (A^2\ C\ A^2BC) \ldots$

A permutes the 4 cyclic G_9 in G^1_{72}

$$\because\ A^{-1}PA = P^xU$$

$$A^{-2}PA^2 = P \text{ or } PA^2C$$

$$\because\ x^2 \equiv 1 \pmod 9$$

U is an operation in $\{B, C\}$, $\{A^2, C\}$ or $\{A^2B, BC\}$ according to which isomorphism we choose as J_P.

Since $\{A^2, B, C\}$ is self-conjugate in $G_{16.9}$, the isomorphisms answering to A and P satisfy

$$J_P J_A = J_A J_P x \quad \ldots \ldots \quad (2)$$

This relation is only satisfied by $x \equiv 1$. Whichever operation is chosen as U we always obtain

$$A^{-2}PA^2 = P$$

Four isomorphic groups are obtained, which all contain a self-conjugate G^4_{16}. In order to shew this, AB, AC or ABC are chosen instead of A as new generating operations.

(*b*) G^5_{16} contains 15 G_8, any one of which may be chosen

$$\because\ J_P = (A)\ (B\ C\ BC) \ldots$$

Hence it follows that

$$D^{-1}PD = P^xB^yC^z$$

Since D^2 and P are permutable and J_D is the identical isomorphism of $\{A, B, C\}$, it follows that $x \equiv 1$. For different values of y and z we obtain four groups which are isomorphic and have a self-conjugate G^5_{16}. D is exchanged for DB, DC or DBC.

(*c*) G^{9}_{16} contains two Abelian G_8 of type (1, 1, 1). As our self-conjugate G_8 we can choose $\{A^2, B, C\}$. P is permutable with an operation in this G_8. For this we may choose A^2, B or C. P is, moreover, permutable with a G_4 which does not contain the operation permutable with P.

$$\because J_P = (A^2)\ (B\ C\ BC) \ldots,\ (B)\ (A^2\ C\ A^2C) \ldots,$$
$$(B)\ (A^2\ BC\ A^2BC) \ldots,\ (B)\ (A^2\ C\ A^2BC) \ldots,$$
$$(C)\ (A^2\ B\ A^2B) \ldots \text{ or } (C)\ (A^2\ B\ A^2BC) \ldots$$
$$\because A^{-1}PA = P^xU_1$$

U_1 is an operation in G_8 which must be chosen so that

$A^{-2}PA^2 = PU_2$ where U_2 is determined by the isomorphism that was chosen for J_P

$$\because x^2 \equiv 1 \pmod 9$$

The isomorphisms corresponding to A and P satisfy the relation (2) only when $x \equiv -1$ and for $J_P = (C)\ (A^2\ B\ A^2B) \ldots$ U_1 is an operation in $\{A^2, B\}$ and it is, moreover, so chosen that
$$A^{-2}PA^2 = PA^2B$$
$$\because U_1 = A^2B \text{ or } A^2$$

By exchanging P and B for P^2 and A^2B we shew that the groups are isomorphic. We thus obtain a single type[1] $G_{16.9}$ defined by the relations

$$A^4 = B^2 = C^2 = P^9 = 1$$
$$B^{-1}AB = A^3 \qquad AC = CA \qquad BC = CB \qquad PC = CP$$
$$P^{-1}A^2P = P \qquad P^{-1}BP = A^2B \qquad A^{-1}PA = P^{-1}A^2B$$

(*d*) G^{10}_{16}. As our self-conjugate G_8 we choose $\{A^2, B, C\}$

$$\because J_P = (A^2)\ (B\ C\ BC) \ldots,\ (A^2)\ (B\ C\ A^2BC) \ldots,$$
$$(C)\ (A^2\ B\ A^2B) \ldots,\ (B)\ (A^2\ C\ A^2C) \ldots,$$
$$(B)\ (A^2\ C\ A^2BC) \ldots \text{ or } (B)\ (A^2\ BC\ A^2C) \ldots$$
$$\because A^{-1}PA = P^xU_1$$

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[1] $G_{16.9}$ is the direct product of $\{C\}$ and $G_{72} = \{A, B, P\}$.

U_1 is an operation in G_8 chosen so that

$$A^{-2}PA^2 = PU_2 \text{ where } U_2 \text{ is determined}$$

by the isomorphism chosen for J_P

$$\therefore\ x^2 \equiv 1 \pmod 9$$

The relation (2) is satisfied by the isomorphisms corresponding to A and P only when $x \equiv -1$ and for $J_P = (A^2)(B\ C\ BC) \ldots$

U_1 is an operation in $\{B, C\}$ and is, moreover, so chosen that

$$A^{-2}PA^2 = P$$

$$\therefore\ U_1 = E \text{ or } BC$$

Both lead to the same group. We thus obtain a single type $G_{16\cdot 9}$ defined by the relations

$$A^4 = B^2 = C^2 = P^9 = 1 \quad B^{-1}AB = AC \quad AC = CA \quad BC = CB$$

$$P^{-1}BP = C \quad P^{-1}CP = BC \quad A^{-1}PA = P^{-1}$$

G^2_{72}.

This group can appear as a self-conjugate sub-group of $G_{16\cdot 9}$ only when the conjugated sequence of groups G_{16} to $G_{16\cdot 9}$ is one of the types G^7_{16}, G^{11}_{16}, G^{13}_{16} or G^{14}_{16}. The quaternion-group $\{T_1, T_2\}$ is self-conjugate in $G_{16\cdot 9}$.

(*a*) G^7_{16}. As our self-conjugate G_8 we choose $\{AC, BC\}$. P is permutable with C^2 and permutes the 3 G_4 in G_8 cyclically. We can suppose without limitation $J_P = (C^2)(AC\ BC\ ABC^2) \ldots$

C permutes the 4 cyclic G_9 in G^2_{72}.

$$\therefore\ C^{-1}PC = P^xU$$

$U(\neq C^2)$ is an operation in $G_8 = \{AC, BC\}$ so chosen that

$$C^{-2}PC^2 = P$$

$$\therefore\ x^2 \equiv 1 \pmod 9$$

G_8 being self-conjugate in $G_{16.9}$, the isomorphisms corresponding to C, U and P must satisfy

$$J_P J_C = J_C J_P{}^x J_U$$

Since J_C is the identical and J_U an inner isomorphism of G_8, this relation can be satisfied only for $x \equiv 1$ and $U = E$. The group obtained contains a self-conjugate G^{7}_{16}.

(b) G^{11}_{16}. This is the direct product of the quaternion-group $\{A, B\}$ and $\{C\}$. We can suppose $J_P = (A^2)\ (A\ B\ AB) \ldots$

$$\therefore\ C^{-1}PC = P^x U \qquad U \neq A^2$$

As before it is proved that $x \equiv 1$ and $U = E$. The group contains a self-conjugate G^{11}_{16}.

(c) G^{13}_{16}. As our self-conjugate G_8 we choose $\{A^2, BA\}$. P is permutable with A^4 and permutes the $3\ G_4$ in G_8 cyclically. We may suppose without limitation $J_P = (A)\ (A^2\ BA\ BA^7)$

$$\therefore\ A^{-1}PA = P^x U$$

$U (\neq A^4)$ is an operation in G_8 so chosen

that $$A^{-2}PA^2 = PBA^7 \qquad \ldots\ldots\ (3)$$

$$\therefore\ x^2 \equiv 1 \pmod 9$$

$x \equiv 1$ $$\therefore\ A^{-1}PA = PU$$

$$A^{-2}PA^2 = PUA^{-1}UA$$

$U = E$, A^2, BA or BA^7 which do not satisfy (3)

$x \equiv -1$ $$\therefore\ A^{-1}PA = P^{-1}U$$

$$A^{-2}PA^2 = U^{-1}PA^{-1}UA$$

$U = E$, A^6, BA^5 or BA^3, of which only the last satisfies (3) and at the same time gives an isomorphism of G^{2}_{72}. We obtain a single type $G_{16.9}$, defined by the relations

$$A^8 = B^2 = P^9 = 1 \qquad B^{-1}AB = A^3$$

$$P^{-1}A^2P = BA \qquad P^{-1}BAP = BA^7 \qquad A^{-1}PA = P^{-1}BA^3$$

(*d*) G^{14}_{16}. As our self-conjugate G_8 we can choose $\{A^2, B\}$ P is permutable with A^4 and permutes the 3 G_4 in G_8 cyclically. We may suppose without limitation

$$(J_P = (A^4)\ (A^2\ \ B\ \ A^2B)\ .\ .\ .$$

$$\because\ A^{-1}PA = P^xU$$

$U(\neq A^4)$ is an operation in G_8 so chosen

that $$A^{-2}PA^2 = PBA^6 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

$$\because\ x^2 \equiv 1 \pmod 9$$

$x \equiv 1$ $$\because\ A^{-2}PA^2 = PUA^{-1}UA$$

$U = E,\ A^2,\ B$ or A^2B which do not satisfy (4)

$x \equiv -1$ $$\because\ A^{-2}PA^2 = U^{-1}PA^{-1}UA$$

$U = E,\ A^6,\ BA^4$ or A^6B, of which values only $U = A^6$ satisfies (4)

We get a single type $G_{16.9}$, defined by the relations

$$A^8 = B^4 = P^9 = 1 \qquad A^4 = B^2 \qquad B^{-1}AB = A^7$$

$$P^{-1}A^2P = B \qquad P^{-1}BP = A^2B \qquad A^{-1}PA = P^{-1}A^6$$

G_9 *non-cyclic*

As our self-conjugate sub-groups G_{72} we may suppose (Page 37):

$$G^3_{72} = \left\{ \begin{array}{c} T_1^2 = T_2^2 = T_3^2 = P_1^3 = P_2^3 = 1 \\ P_1P_2 = P_2P_1 \quad T_1T_2 = T_2T_1 \quad T_1T_3 = T_3T_1 \quad T_2T_3 = T_3T_2 \\ P_2^{-1}T_1P_2 = T_1 \quad P_2^{-1}T_2P_2 = T_3 \quad P_2^{-1}T_3P_2 = T_2T_3 \\ P_1T_1 = T_1P_1 \quad P_1T_2 = T_2P_1 \quad P_1T_3 = T_3P_1 \end{array} \right\}$$

$$G^4_{72} = \left\{ \begin{array}{c} T_1^4 = T_2^4 = P_1^3 = P_2^3 = 1 \\ P_1P_2 = P_2P_1 \quad T_1^2 = T_2^2 \quad T_2^{-1}T_1T_2 = T_1^3 \\ P_1T_1 = T_1P_1 \quad P_1T_2 = T_2P_1 \\ P_2^{-1}T_1P_2 = T_2 \quad P_2^{-1}T_2P_2 = T_1T_2 \end{array} \right\}$$

$$G^5_{72} = \begin{cases} T_1^4 = T_2^2 = P_1^3 = P_2^3 = 1 \\ P_1P_2 = P_2P_1 \quad T_2^{-1}T_1T_2 = T_1^3 \quad P_1T_1 = T_1P_1 \quad P_1T_2 = T_2P_1 \\ T_1^{-1}P_2T_1 = P_2^{-1}T_1^2T_2 \\ P_2^{-1}T_1^2P_2 = T_2 \qquad P_2^{-1}T_2P_2 = T_1^2T_2 \end{cases}$$

G^3_{72}.

(*a*) G^4_{16}. $\because J_P = (A^2)\ (B\ \ C\ \ BC) \ldots,\ (B)\ (A^2\ \ C\ \ A^2C) \ldots$ or $(B)\ (A^2\ \ C\ \ A^2BC) \ldots$

A permutes the 4 non-cyclic G_9 in G^3_{72}.

$$\because\ A^{-1}P_1A = P_1^x \qquad A^{-1}P_2A = P_1^yP_2^zU$$

U is an operation in $\{B,\ C\}$, $\{A^2,\ C\}$ or $\{A^2B,\ BC\}$ according to which isomorphism we choose as J_{P_2}

Since $\qquad A^{-2}P_1A^2 = P_1 \quad A^{-2}P_2A^2 = P_2$ or P_2A^2C

we get $\qquad x^2 \equiv z^2 \equiv 1 \quad y(x+z) \equiv 0 \pmod 3$

$G_8 = \{A^2,\ B,\ C\}$ being self-conjugate in $G_{16.9}$, the isomorphisms corresponding to A, P_1 and P_2 satisfy the relations

$$\left.\begin{aligned} J_{P_1}J_A &= J_AJ_{P_1}^x \\ J_{P_2}J_A &= J_AJ_{P_1}^yJ_{P_2}^z \end{aligned}\right\} \quad \ldots\ldots \quad (5)$$

Hence it follows that $\qquad z \equiv 1$

$$\because \begin{cases} x \equiv 1 \\ y \equiv 0 \end{cases} \text{or} \begin{cases} x \equiv -1 \\ y \equiv 0,\ 1,\ 2 \end{cases}$$

Since $z \equiv 1 \qquad A^{-2}P_2A^2 = P_2U^2 \qquad$ and hence it follows that $J_{P_2} = (A^2)\ (B\ \ C\ \ BC) \ldots$

$x \equiv 1$. U is an operation in $\{B,\ C\}$. The four groups are isomorphic and all contain a self-conjugate G^4_{16}.

$x \equiv -1$. Independently of y we can choose $U = E$. If we then exchange P_2 for $P_1^tP_2$, where $2t + y \equiv 0 \pmod 3$, it appears that the groups are isomorphic. We obtain single type $G_{16.9}$, defined by the relations

$$A^4 = B^2 = C^2 = P_1^3 = P_2^3 = 1$$

$$AB = BA \qquad AC = CA \qquad BC = CB \qquad P_1P_2 = P_2P_1$$

$$A^{-1}P_1A = P_1^{-1} \qquad P_1B = BP_1 \qquad P_1C = CP_1$$

$$P_2A = AP_2 \qquad P_2^{-1}BP_2 = C \qquad P_2^{-1}CP_2 = BC$$

(b) G^5_{16}. $\quad \because J_{P_2} = (A)\ (B\ C\ BC) \ldots$

$$\because D^{-1}P_1D = P_1^x \qquad D^{-1}P_2D = P_1^yP_2^zU \qquad U \text{ in } \{B, C\}$$

Since D^2 is permutable with P_1 and P_2 and J_D is the identical isomorphism of $\{A, B, C\}$, we get

$$z \equiv 1, \quad \begin{cases} x \equiv 1 \\ y \equiv 0 \end{cases} \text{ or } \begin{cases} x \equiv -1 \\ y \equiv 0, 1, 2 \end{cases}$$

$x \equiv 1$. The four groups are isomorphic and contain a self-conjugate G^5_{16}.

$x \equiv -1$. We obtain a single type[1] defined by the relations

$$A^2 = B^2 = C^2 = D^2 = P_1^3 = P_2^3 = 1 \qquad AB = BA \qquad AC = CA$$

$$AD = DA \qquad BC = CB \qquad BD = DB \qquad CD = DC \qquad P_1P_2 = P_2P_1$$

$$AP_1 = P_1A \qquad BP_1 = P_1B \qquad CP_1 = P_1C \qquad D^{-1}P_1D = P_1^{-1}$$

$$AP_2 = P_2A \qquad P_2^{-1}BP_2 = C \qquad P_2^{-1}CP_2 = BC \qquad DP_2 = P_2D$$

(c) G^9_{16} $\quad \because J_{P_2} = (A^2)\ (B\ C\ BC) \ldots, \ (B)\ (A^2\ C\ A^2C) \ldots,$
$(B)\ (A^2\ BC\ A^2BC) \ldots, \quad (B)\ (A^2\ C\ A^2BC) \ldots,$
$(C)\ (A^2\ B\ A^2B) \ldots$ or $(C)\ (A^2\ B\ A^2BC) \ldots$

$$A^{-1}P_1A = P_1^x \qquad A^{-1}P_2A = P_1^yP_2^zU_1$$

U_1 is an operation in $\{A^2, B, C\}$ which must be so chosen that $\quad A^{-2}P_2A^2 = P_2U_2$ where U_2 is determined by the isomorphism chosen for J_{P_2}

$$\because x^2 \equiv z^2 \equiv 1 \qquad y(x + z) \equiv 0 \pmod 3 \quad \ldots \quad (6)$$

---

[1] $\{D, B, C, P_1, P_2\}$ is a group of order 72 which lacks both self-conjugate G_8 and G_9. The defining relations for this are not given in Tripp's investigation.

The isomorphisms corresponding to A, P_1 and P_2 satisfy (5) only when $z = -1$ and for $J_{P_2} = (C)\ (A^2\ B\ A^2B)\ \ldots$ U_1 is an operation in $\{A^2, B\}$ and is, moreover, so chosen that

$$A^{-2}P_2A^2 = P_2A^2B$$

$$\because U_1 = A^2 \text{ or } A^2B$$

From (6) we get

$$\begin{cases} x = 1 \\ y = 0,\ 1,\ 2 \end{cases} \text{or} \begin{cases} x = -1 \\ y = 0 \end{cases}$$

$x = 1$. The groups are isomorphic. We thus obtain a single type[1] $G_{16.9}$, defined by the relations

$$A^4 = B^2 = C^2 = P_1^3 = P_2^3 = 1$$

$$B^{-1}AB = A^3 \qquad AC = CA \qquad BC = CB \qquad P_1P_2 = P_2P_1$$

$$A^{-1}P_2A = P_2^{-1}A^2B$$

$$P_2^{-1}A^2P_2 = B \qquad P_2^{-1}BP_2 = A^2B \qquad P_2C = CP_2$$

$$P_1A = AP_1 \qquad P_1B = BP_1 \qquad P_1C = CP_1$$

$x = -1$. We get a single type[2] $G_{16.9}$, defined by the relations

$$A^4 = B^2 = C^2 = P_1^3 = P_2^3 = 1$$

$$B^{-1}AB = A^3 \qquad AC = CA \qquad BC = CB \qquad P_1P_2 = P_2P_1$$

$$A^{-1}P_2A = P_2^{-1}A^2B$$

$$P_2^{-1}A^2P_2 = B \qquad P_2^{-1}BP_2 = A^2B \qquad P_2C = CP_2$$

$$A^{-1}P_1A = P_1^{-1} \qquad P_1B = BP_1 \qquad P_1C = CP_1$$

(*d*) G_{16}^{10} $\because J_{P_2} = (A^2)\ (B\ C\ BC)\ldots,\ (A^2)\ (B\ C\ A^2BC)\ldots,$ $(C)\ (A^2\ B\ A^2B)\ldots,\ (B)\ (A^2\ C\ A^2C)\ldots.\ (B)\ (A^2\ C\ A^2BC)\ldots$ or $(B)\ (A^2\ BC\ A^2C)\ \ldots$

$$\because A^{-1}P_1A = P_1^x \qquad A^{-1}P_2A = P_1^yP_2^zU_1$$

---

[1] $G_{16.9}$ is the direct product of $\{C\}$, $\{P_1\}$ and $G_{24} = \{A, B, P_2\}$.

[2] This type is the direct product of $\{C\}$ and $G_{72} = \{A, B, P_1, P_2\}$.

U_1 is an operation in $\{A^2, B, C\}$ so chosen that

$A^{-2}P_2A^2 = P_2U_2$ where U_2 is determined by the isomorphism that was chosen for J_{P_2}

$$\because\ x^2 \equiv z^2 \equiv 1 \quad y(x+z) \equiv 0 \pmod 3 \quad . \quad . \ (7)$$

The relations (5) are satisfied by the isomorphisms corresponding to A, P_1 and P_2 only when $z \equiv -1$ and for $J_{P_2} = (A^2)\ (B\ C\ BC)\ .\ .\ .$

U_1 is an operation in $\{B, C\}$ and is, moreover, so chosen that

$$A^{-2}P_2A^2 = P_2$$

$$\because\ U_1 = E \text{ or } BC$$

From (7) we get

$$\begin{cases} x \equiv 1 \\ y \equiv 0,\ 1,\ 2 \end{cases} \text{ or } \begin{cases} x \equiv -1 \\ y \equiv 0 \end{cases}$$

$x \equiv 1$. The groups are isomorphic, We thus obtain a single type[1] $G_{16\cdot 9}$, defined by the relations

[1] Groups G_{16p}, which lack both self-conjugate G_{16} and G_p. only exist for $p = 3$, 5 or 7. When $p = 5$ or 7, it is shewn that no groups exist. This can be proved, for instance, in the same way as for G_{16p^2} (cyclic G_{p^2}). For since the groups $G_{16\cdot 7}$ are soluble they have a self-conjugate G_{56} which necessarily contains an Abelian G_8 of type (1, 1, 1) self-conjugate.

Thus there remains only $p = 3$. Levavasseurs investigation is in this case incorrect. He sets up the defining relations for only two groups, while in reality there are four G_{48}. These G_{48} have 3 G_{16} and consequently a self-conjugate G_{24} which in its turn has a self-conjugate G_8. The different types G_{48} are thus obtained directly in the same way as the corresponding $G_{16\cdot 9}$ (cyclic G_9).

The direct product of such a group G_{48} with a cyclic G_3 give a group $G_{16\cdot 9}$ belonging here.

$\{A, B, C, P_2\}$ is a group of order 48, which lack both self-conjugate G_{16} and G_3. The defining relations for this are not given in Levavasseurs investigation. As permutation-group of degree 16, it is given by G. Bolinder, Über die Strukturverhältnisse bei einer besonderen Klasse vollkommener Gruppen (Diss. Uppsala 1909 Page 120). We may take, for instance,

$$A = (boje)(dmlg)(fphn)(ik), \quad B = (ak)(bl)(ci)(dj)(eo)(fp)(gm)(hn),$$
$$C = (ac)(bd)(eg)(fh)(ik)(jl)(mo)(np) \text{ and } P_2 = (bmn)(ikc)(chd)(ofl)(gpj)$$

$$A^4 = B^2 = C^2 = P_1^3 = P_2^3 = 1$$

$$B^{-1}AB = AC \qquad AC = CA \qquad BC = CB \qquad P_1P_2 = P_2P_1$$

$$A^{-1}P_2A = P_2^{-1} \qquad P_2^{-1}BP_2 = C \qquad P_2^{-1}CP_2 = BC$$

$$AP_1 = P_1A \qquad BP_1 = P_1B \qquad CP_1 = P_1C$$

$x \equiv -1$. We get a single type $G_{16.9}$, defined by the relations

$$A^4 = B^2 = C^2 = P_1^3 = P_2^3 = 1$$

$$B^{-1}AB = AC \qquad AC = CA \qquad BC = CB \qquad P_1P_2 = P_2P_1$$

$$A^{-1}P_2A = P_2^{-1} \qquad P_2^{-1}BP_2 = C \qquad P_2^{-1}CP_2 = BC$$

$$A^{-1}P_1A = P_1^{-1} \qquad BP_1 = P_1B \qquad CP_1 = P_1C$$

G^4_{72}.

(a) G^7_{16}. $\qquad \because J_{P_2} = (C^2)\ (AC \ \ BC \ \ ABC^2) \ldots$

$$\because C^{-1}P_1C = P_1^x \qquad C^{-1}P_2C = P_1^yP_2^zU$$

$U (\neq C^2)$ is an operation in $\{AC, BC\}$ so chosen that

$$C^{-2}P_2C^2 = P_2$$

$$\because x^2 \equiv z^2 \equiv 1 \qquad y(x+z) \equiv 0) \qquad . \quad . \quad . \quad . \quad (8)$$

Since $\{AC, BC\}$ is self-conjugate in $G_{16.9}$, the isomorphisms corresponding to C, U, P_1 and P_2 satisfy

$$J_{P_2}J_C = J_CJ_{P_1}^yJ_{P_2}^zJ_U$$

$J_C = J_{P_1} = J_E \qquad \because z \equiv 1 \qquad U = E$

From (8) we get

$$\begin{cases} x \equiv 1 \\ y \equiv 0 \end{cases} \text{ or } \begin{cases} x \equiv -1 \\ y \equiv 0,\ 1,\ 2 \end{cases}$$

$x \equiv 1$. The group contains a self-conjugate G^7_{16}.

$x \equiv -1$. The groups are isomorphic and we obtain a single type defined by the relations

$$A^2 = B^2 = C^4 = P_1^3 = P_2^3 = 1$$

$$B^{-1}AB = AC^2 \qquad AC = CA \qquad BC = CB \qquad P_1P_2 = P_2P_1$$

$$P_2^{-1}ACP_2 = BC \qquad P_2^{-1}BCP_2 = ABC^2 \qquad CP_2 = P_2C$$
$$AP_1 = P_1A \qquad BP_1 = P_1B \qquad C^{-1}P_1C = P_1^{-1}$$

(*b*) G^{11}_{16}. $\quad \because J_{P_2} = (A^2)\ (A\ \ B\ \ AB)\ .\ .\ .$
$$\because C^{-1}P_1C = P_1^{x} \qquad C^{-1}P_2C = P_1^{y}P_2^{z}U$$

$U(\neq A^2)$ is an operation in $\{A, B\}$ so chosen that
$$C^{-2}P_2C^2 = P_2$$

As before we shew that

$U = E,\ z \equiv 1$
$$\because \begin{cases} x \equiv 1 \\ y \equiv 0 \end{cases} \text{ or } \begin{cases} x \equiv -1 \\ y \equiv 0,\ 1,\ 2 \end{cases}$$

$x = 1$. The group contains a self-conjugate G^{11}_{16}.
$x \equiv -1$. The groups are isomorphic and we obtain a single type $G_{16\cdot 9}$, defined by the relations
$$A^4 = B^4 = C^2 = P_1^{3} = P_2^{3} = 1$$
$$A^2 = B^2 \qquad B^{-1}AB = A^3 \qquad AC = CA \qquad BC = CB$$
$$AP_1 = P_1A \qquad BP_1 = P_1B \qquad C^{-1}P_1C = P_1^{-1} \qquad P_1P_2 = P_2P_1$$
$$P_2^{-1}AP_2 = B \qquad P_2^{-1}BP_2 = AB \qquad P_2C = CP_2$$

(*c*) G^{13}_{16}. $\quad \because J_{P_2} = (A^4)\ (A^2\ \ BA\ \ BA^7)\ .\ .\ .$
$$A^{-1}P_1A = P_1^{x} \qquad A^{-1}P_2A = P_1^{y}P_2^{z}U$$

$U(\neq A^4)$ is an operation in $\{A^2, BA\}$ so chosen that
$$A^{-2}P_2A^2 = P_2BA^7 \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$
$$\because x^2 \equiv z^2 \equiv 1 \qquad y(x+z) \equiv 0 \pmod 3 \quad . \quad . \quad (10)$$

$z \equiv 1 \qquad \because A^{-2}P_2A^2 = P_2UA^{-1}UA$

$U = E,\ A^2,\ BA$ or BA^7 which do not satisfy (9)

$z \equiv -1 \qquad \because A^{-2}P_2A^2 = U^{-1}P_2A^{-1}UA$

$U = E,\ A^6,\ BA^5$ or BA^3 of which only $U = BA^3$ satisfies (9) and at the same time gives an isomorphism of G^{4}_{72}.

From (10) we get

$$\begin{cases} x \equiv 1 \\ y \equiv 0,\ 1,\ 2 \end{cases} \text{ or } \begin{cases} x \equiv -1 \\ y \equiv 0 \end{cases}$$

$x \equiv 1$. The groups are isomorphic and we get a single type[1], defined by the relations

$$A^8 = B^2 = P_1^3 = P_2^3 = 1 \qquad B^{-1}AB = A^3 \qquad P_1P_2 = P_2P_1$$
$$P_2^{-1}A^2P_2 = BA \qquad P_2^{-1}BAP_2 = BA^7 \qquad A^{-1}P_2A = P_2^{-1}BA^3$$
$$AP_1 = P_1A \qquad BP_1 = P_1B$$

$x \equiv -1$. This gives a new type, defined by the relations

$$A^8 = B^2 = P_1^3 = P_2^3 = 1 \qquad B^{-1}AB = A^3 \qquad P_1P_2 = P_2P_1$$
$$P_2^{-1}A^2P_2 = BA \qquad P_2^{-1}BAP_2 = BA^7 \qquad A^{-1}P_2A = P_2^{-1}BA^3$$
$$A^{-1}P_1A = P_1^{-1} \qquad B^{-1}P_1B = P_1$$

(*d*) G^{14}_{16}. $\quad \because J_{P_2} = (A^4)\ (A^2\ B\ A^2B)\ \ldots$

$\because A^{-1}P_1A = P_1^x \qquad A^{-1}P_2A = P_1^yP_2^zU$

$U(\neq A^4)$ is an operation in $\{A^2, B\}$ so chosen that

$$A^{-2}P_2A^2 = P_2BA^6 \quad \ldots\ldots \quad (11)$$

$$\because x^2 \equiv z^2 \equiv 1 \quad y(x+z) \equiv 0 \pmod 3) \quad . \quad . \quad (12)$$

$z \equiv 1 \qquad \because A^{-2}P_2A^2 = P_2UA^{-1}UA$

$U = E$, A^2, B or A^2B which do not satisfy (11)

$z \equiv -1 \qquad \because A^{-2}P_2A^2 = U^{-1}P_2A^{-1}UA$

$U = E$, A^6, BA^4 or A^6B of which values only $U = A^6$ satisfies (11)

[1] $\{A, B, P_2\}$ is a group of order 48 isomorphic with the group of isomorphisms of the non-cyclic G_9. We may take, for instance, $A = \begin{pmatrix} -1, & 1 \\ -1, & -1 \end{pmatrix}$, $B = \begin{pmatrix} 1, & 0 \\ 0, & -1 \end{pmatrix}$ and $P_2 = \begin{pmatrix} 1, & 1 \\ 0, & 1 \end{pmatrix}$. Burnside, Theory of groups of finite order. Cap. 20 (1911).

From (12) we get

$$\begin{cases} x \equiv 1 \\ y \equiv 0,\ 1,\ 2 \end{cases} \quad \text{or} \quad \begin{cases} x \equiv -1 \\ y \equiv 0 \end{cases}$$

$x \equiv 1$. The groups are isomorphic and we get a single type[1] $G_{16.9}$, defined by the relations

$$A^8 = B^4 = P_1^3 = P_2^3 = 1$$
$$A^4 = B^2 \qquad B^{-1}AB = A^7 \qquad P_1P_2 = P_2P_1$$
$$P_2^{-1}A^2P_2 = B \qquad P_2^{-1}BP_2 = A^2B \qquad A^{-1}P_2A = P_2^{-1}A^6$$
$$P_1A = AP_1 \qquad P_1B = BP_1$$

$x \equiv -1$. This gives a new type, defined by the relations

$$A^8 = B^4 = P_1^3 = P_2^3 = 1$$
$$A^4 = B^2 \qquad B^{-1}AB = A^7 \qquad P_1P_2 = P_2P_1$$
$$P_2^{-1}A^2P_2 = B \qquad P_2^{-1}BP_2 = A^2B \qquad A^{-1}P_2A = P_2^{-1}A^6$$
$$A^{-1}P_1A = P_1^{-1} \qquad B^{-1}P_1B = P_1$$

G^5_{72}.

This group can appear as a self-conjugate sub-group of $G_{16.9}$ only when the conjugated sequence of groups G_{16} to $G_{16.9}$ is one of the types G^7_{16}, G^9_{16}, G^{12}_{16} or G^{13}_{16}.

(*a*) G^7_{16}. As our dihedral group we can choose $\{AC, B\}$. By varying the generating operations we can without limitation suppose

$$P_2^{-1}C^2P_2 = B \qquad P_2^{-1}BP_2 = BC^2 \qquad (AC)^{-1}P_2AC = P^{-1}C^2B$$
$$\because\ C^{-1}P_1C = P_1^x \qquad C^{-1}P_2C = P_1^yP_2^zU$$

U is an operation in $\{B, C^2\}$, and is so chosen that

$$C^{-2}P_2C^2 = P_2C^2B$$

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[1] $\{A, B, P_2\}$ is a group of order 48 which lacks both self-conjugate G_{16} and G_3. The defining relations for this are not given in Levavasseur's investigation.

When $z \equiv -1$ this relation is satisfied for $U = C^2$. J_C defined in this way does not give an isomorphism of G^5_{72}. Consequently there is no group.

(b) G^9_{16} is the direct product of the dihedral group $\{A, B\}$ and $\{C\}$ and we can suppose

$$P_2^{-1}A^2P_2 = B \qquad P_2^{-1}BP_2 = A^2B \qquad A^{-1}P_2A = P_2^{-1}A^2B$$

$$\therefore\ C^{-1}P_1C = P_1^x \quad C^{-1}P_2C = P_1^yP_2^zU \quad \text{where } U \text{ is}$$

so chosen that $\qquad C^{-2}P_2C^2 = P_2$

$$\therefore\quad x^2 \equiv z^2 \equiv 1 \qquad y(x+z) \equiv 0$$

$z \equiv -1 \qquad\qquad \therefore\ U = E$

J_C defined in this way does not give an isomorphism of G^5_{72}

$z \equiv 1 \qquad\qquad \therefore\ y \equiv 0, \quad U = E$ or A^2B

When $x \equiv 1$ the groups have also a self-conjugate G^3_{72}. For $x \equiv -1$ we get a new type, defined by the relations

$$A^4 = B^2 = C^2 = P_1^3 = P_2^3 = 1$$

$$B^{-1}AB = A^3 \qquad AC = CA \qquad BC = CB \qquad P_1P_2 = P_2P_1$$

$$P_2^{-1}A^2P_2 = B \qquad P_2^{-1}BP_2 = A^2B \qquad A^{-1}P_2A = P_2^{-1}A^2B$$

$$P_1A = AP_1 \qquad BP_1 = P_1B \qquad C^{-1}P_1C = P_1^{-1} \qquad CP_2 = P_2C$$

(c) G^{12}_{16} and G^{13}_{16} contain the dihedral group $\{A^2, B\}$. In both cases we can without limitation suppose

$$P_2^{-1}A^4P_2 = B \qquad P_2^{-1}BP_2 = A^4B \qquad A^{-2}P_2A^2 = P_2^{-1}A^4B$$

$$\therefore\ A^{-1}P_2A = P_1^yP_2^zU$$

Since $\qquad A^{-2}P_2A^2 = P_2^{-1}A^4B$

We get $\qquad z^2 \equiv -1 \pmod 3$, which has no solution. Consequently there is no group.

Finally I give a table showing the number of types for different values of p

p	Number of types
3	197
5	221
7	172
$p \equiv 1 \pmod{16}$	257
$p \equiv 3 \pmod{8}$	167
$p \equiv 5 \pmod{8}$	219
$p \equiv 7 \pmod{8}$	169
$p \equiv 9 \pmod{16}$	243

The groups of order 8 p^3.

Sylow's theorem shows that, with the exception of $p = 3$ and 7, all the investigated groups G_{8p^3} contain a self-conjugate G_{p^3}.

I.

The G_{8p^3} which have a self-conjugate G_8 and also a self-conjugate G_{p^3}.

The sub-groups G_{p^3} and G_8 have no common operation except the identity. Every operation in G_8 must thus be permutable with every operation in G_{p^3}. The investigated groups G_{8p^3} are therefore obtained as the direct product of 1 G_8 and 1 G_{p^3}[1]. 25 groups are obtained, 9 of which are Abelian. The sub-groups of the direct product af the quaternion-group and an Abelian G_{p^3} are all self-conjugate. These three groups are the only G_{8p^3} except the Abelian ones which have this property.

II.

The G_{8p^3} which have a self-conjugate G_{p^3} and more than one G_8.

The factor-group G_{8p^3}/G_{p^3} is isomorphic with any one of the 5 types G_8. The operations in G_8 which are permutable with every operation in G_{p^3} form a self-conjugate sub-group H of G_8. To every operation in the factor-group G_8/H there thus corresponds an isomorphism of G_{p^3}. The setting up of

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[1] Burnside, Theory of groups of finite order (1911) Cap. 10.

the different G_{8p^3} then proceeds according to the type of H, which considerably simplifies the process. As G_8/H is a cyclic group, the results are obtained directly, since the G_{8p^3} which have a self-conjugate G_{p^3} and more than one cyclic G_8 have already been produced. Two groups H, which cannot be transferred one into the other by changing the generating operations of G_8, give distinct groups.

$$G^1_{p^3} = \{P^{p^3} = 1\}$$

The group of isomorphisms[1] is cyclic. The factor-group G_8/H is thus necessarily cyclic. The results are set forth in the following table

$G_{8p^3}/G^1_{p^3}$	H	$A^{-1}PA$	$B^{-1}PB$	$C^{-1}PC$	Aritm. Rel.
G^1_8	$\{A^2\}$	P^{-1}			
	$\{A^4\}$	P^α			$p \equiv 1 \pmod 4$
	E	P^α			$p \equiv 1 \pmod 8$
G^2_8	$\{A\}$	P	P^{-1}		
	$\{A^2, B\}$	P^{-1}	P		
	$\{B\}$	P^α	P		$p \equiv 1 \pmod 4$
G^3_8	$\{A, B\}$	P	P	P^{-1}	
G^4_8	$\{A\}$	P	P^{-1}		
	$\{A^2, B\}$	P^{-1}	P		
G^5_8	$\{A\}$	P	P^{-1}		

In every particular case we can take a fixed value of α which satisfies the congruence obtained. The remaining α-values give groups isomorphic with this. All the types in the table are distinct, as appears from the method of arrangement.

[1] Proc. L. M. S. Bd 30 (Pages 211—216).

$$G_{p^3}^2 = \{P_1^{p^2} = P_2^p = 1 \qquad P_1P_2 = P_2P_1\}$$

$$G_{8p^3}/G_{p^3}^2 = \{A^8 = 1\}$$

$G_{p^3}^2$ has p cyclic sub-groups G_{p^2}, viz. $\{P_1P_2^k\}$ $(k = 0, 1, \ldots p-1)$, of which one at least is permutable with A. For this we can choose $\{P_1\}$. Besides the sub-group $\{P_1^p\}$ one of the groups $\{P_1^{kp}P_2\}$ $(k = 0, 1, \ldots p-1)$ must also be permutable with A and for this we can choose $\{P_2\}$

We have $\qquad J_A = (P_1, P_2; P_1^\alpha, P_2^\beta)$

$$\because \alpha^8 \equiv 1 \pmod{p^2} \qquad \beta^8 \equiv 1 \pmod{p}$$

$H = \{A^2\} \qquad \because J_A = (P_1, P_2; P_1^{\pm 1}, P_2^{\pm 1})$. Three types

$H = \{A^4\} \qquad \because \alpha^4 \equiv 1 \pmod{p^2} \qquad \beta^4 \equiv 1 \pmod{p}$

Consequently there are three cases to notice:

(1) $\alpha \equiv \pm 1$, β a fixed prim. root of $\beta^4 \equiv 1$

(2) α a fixed prim. root of $\alpha^4 \equiv 1$, β the prim. roots of $\beta^4 \equiv 1$

(3) α a fixed prim. root of $\alpha^4 \equiv 1$, $\beta \equiv \pm 1$

The types are all distinct

$H = E \qquad \because \alpha^8 \equiv 1 \pmod{p^2} \qquad \beta^8 \equiv 1 \pmod{p}$

We now have the following three cases:

(1) α the roots of $\alpha^4 \equiv 1$, β a fixed prim. root of $\beta^8 \equiv 1$

(2) α a fixed prim. root of $\alpha^8 \equiv 1$, β the prim. roots of $\beta^8 \equiv 1$

(3) α a fixed prim. root of $\alpha^8 \equiv 1$, β the roots of $\beta^4 \equiv 1$

The types are all distinct.

The group of isomorphisms of $G_{p^3}^2$ is of order $p^3(p-1)^2$. The conjugated sequence of Sylow's sub-groups of order 2^k to this group consists of Abelian groups generated by two operations of order $2^{\frac{k}{2}}$. Consequently the group G_{2^k} contains no Abelian G_8 of type (1, 1, 1) and not more than one non-cyclic G_4.

The results here can thus be easily seen directly.

$G_{8p^3}/G^2_{p^3} = \{A^4 = B^2 = 1 \quad AB = BA\}$

$H = \{A\}$ or $\{A^2, B\}$. Six types.
$H = \{B\}$. Six types for $p \equiv 1 \pmod 4$
$H = \{A^2\}$

$\because J_A = (P_1, P_2; P_1, P_2^{-1}) \qquad J_B = (P_1, P_2; P_1^{-1}, P_2^{\pm 1})$

or $\qquad J_A = (P_1, P_2; P_1^{-1} P_2^{-1}) \qquad J_B = (P_1, P_2; P_1, P_2^{-1})$

$H = E$

$\because J_A = (P_1, P_2; P_1, P_2^{\beta}) \qquad J_B = (P_1, P_2; P_1^{-1}, P_2)$

$J_A = (P_1, P_2; P_1^{\alpha}, P_2^{\beta}) \qquad J_B = (P_1, P_2; P_1^{-1}, P_2)$

or $\qquad J_A = (P_1, P_2; P_2^{\alpha}, P_2) \qquad J_B = (P_1, P_2; P_1, P_2^{-1})$

$G_{8p^3}/G^2_{p^3} = \{A^2 = B^2 = C^2 = 1 \quad AB = BA \quad AC = CA \quad BC = CB\}$

$H = \{A, B\}$. Three types.
$H = \{C\}$

$\because J_A = (P_1, P_2; P_1^{-1}, P_2) \quad J_B = (P_1, P_2; P_1, P_2^{-1})$

$G_{8p^3}/G^2_{p^3} = \{A^4 = B^2 = 1 \quad B^{-1}AB = A^3\}$

$H = \{A\}$ or $\{A^2, B\}$. Six types.
$H = \{A^2\}$

$\because J_A = (P_1, P_2; P_1^{\pm 1}, P_2^{-1}) \qquad J_B = (P_1, P_2; P_1^{-1}, P_2)$

or $\qquad J_A = (P_1, P_2; P_1^{-1}, P_2) \qquad J_B = (P_1, P_2; P_1, P_2^{-1})$

$G_{8p^3}/G^2_{p^3} = \{A^4 = B^4 = 1 \quad A^2 = B^2 \quad B^{-1}AB = A^3\}$

$H = \{A\}$. Three types.
$H = \{A^2\}$

$\because J_A = (P_1, P_2; P_1, P_2^{-1}) \qquad J_B = (P_1, P_2; P_1^{-1}, P_2)$

$G^3_{p^3} = \{P_1^p = P_2^p = P_3^p = 1 \quad P_1P_2 = P_2P_1 \quad P_1P_3 = P_3P_1$
$\quad P_2P_3 = P_3P_2\}$

$G_{8p^3}/G_{p^3}^3 = \{A^8 = 1\}$

$G_{p^3}^3$ contains $p^2 + p + 1$ G_p. Of these r are self-conjugate when transformed by A. The others are permuted in cyclic sequences with 2, 4 or 8 groups in each

$$\because \; p^2 + p + 1 = r + 2k$$

We thus get $\qquad r = 1$ or $r > 3$

(i) $\quad r \geq 3$.

We can then always choose $J_A = (P_1, P_2, P_3; P_1^\alpha, P_2^\beta, P_3^\gamma)$ independently of whether $\alpha \equiv \beta \pmod{p}$ or not.

$H = \{A^2\} \qquad \because \; \alpha^2 \equiv \beta^2 \equiv \gamma^2 \equiv 1 \pmod{p}$

(1) $\alpha \equiv \beta \equiv \gamma \equiv -1$
(2) $\alpha \equiv 1 \quad \beta \equiv \gamma \equiv -1$
(3) $\alpha \equiv \beta \equiv 1 \quad \gamma \equiv -1$

Only these three types are distinct.

$H = \{A^4\} \qquad \because \; \alpha^4 \equiv \beta^4 \equiv \gamma^4 \equiv 1$

(1) α, β and γ the prim. roots of $\delta^4 \equiv 1$. Two types for which $\alpha \equiv \beta \equiv \delta$ and $\gamma \equiv \delta$ or δ^3

(2) α and β the prim. roots of $\delta^4 \equiv 1$, $\gamma = \pm 1$. Four types for which $\alpha \equiv \delta$ and $\beta \equiv \delta$ or δ^3

(3) α the prim. roots of $\delta^4 \equiv 1$, β and $\gamma \equiv \pm 1$. Three types for which $\alpha \equiv \delta$, $\beta \equiv \pm 1$, $\gamma \equiv 1$ and $\alpha \equiv \delta$, $\beta \equiv \gamma \equiv -1$

$H = E \qquad \because \; \alpha^8 \equiv \beta^8 \equiv \gamma^8 \equiv 1$

(1) α, β and γ the prim. roots of $\delta^8 \equiv 1$. Five types for which $\alpha \equiv \beta \equiv \delta$, $\gamma \equiv \delta^r (r = 1, 3, 5, 7)$ and $\alpha \equiv \delta$, $\beta \equiv \delta^3$, $\gamma \equiv \delta^5$

(2) α and β the prim. roots of $\delta^8 \equiv 1$, $\gamma \equiv \pm 1$, δ^2 or δ^6. Fourteen types for which $\alpha \equiv \delta$, $\beta \equiv \delta^r (r = 1, 3, 5, 7)$, $\gamma = \pm 1$ or δ^2 and $\alpha \equiv \delta$, $\beta \equiv \delta^r (r = 1, 5)$, $\gamma \equiv \delta^6$

(3) α the prim. roots of $\delta^8 \equiv 1$, β and $\gamma \equiv \pm 1$, δ^2 or δ^6. Three types for which $\alpha \equiv \delta$, $\beta \equiv \delta^2$, $\gamma \equiv \delta^r (r = 2, 6)$ and $\alpha \equiv \delta$, $\beta \equiv \gamma \equiv \delta^6$. Seven types for which $\alpha \equiv \delta$, $\beta = \delta^r (r = 0, 2, 4, 6)$, $\gamma \equiv \pm 1$ (except $\beta \equiv -\gamma \equiv 1$).

(*ii*) $r=1$

We can choose $J_A=(P_1, P_2, P_3, P_1^a, P_3, P_1^aP_2^bP_3^c)$. If P_2 is transformed into an operation in $\{P_1, P_2\}$ we are brought back to the previous case. The permutation of all G_p except $\{P_1\}$ corresponding to the isomorphism J_A contains cycles with 2, 4 or 8 terms.

1. The permutation of the p^2+pG_p contains one cycle with two terms.

We can then suppose

$$a\equiv c\equiv 0, \qquad b^4\equiv 1 \pmod p$$

When $b\equiv 1$ J_A is an isomorphism for which more than one G_p is self-conjugate. If on the other hand $b\equiv -1$, we obtain for $p\equiv 3 \pmod 4$ two types for which $J_A=(P_1, P_2, P_3; P_1^{\pm 1}, P_3, P_2^{-1})$. If b is a prim. root of $b^4\equiv 1$ we obtain for $p\equiv 5 \pmod 8$ four types for which $J_A=(P_1, P_2, P_3; P_1^{a^r}, P_3, P_2^a)$ $(r=0, 1, 2, 3)$.

2. The permutation contains cycles with at least four terms.

We thus get

$$\left.\begin{aligned} a(a^2+ac+b+c^2)&\equiv 0\\ b(b+c^2)&\equiv \pm 1\\ c(2b+c^2)&\equiv 0 \end{aligned}\right\} \pmod p$$

Because $p^2+p=4k$ it is necessary that $p\equiv 3 \pmod 4$. When $c\equiv 0$ the permutation corresponding to J_A contains at least one cycle with two terms

$$\because\ 2b+c^2\equiv 0$$

Hence it follows that

$$b^2\equiv 1$$

(*i*) $b\equiv 1$ $\qquad \because\ c^2\equiv -2, \quad a\equiv \pm 1, \quad a\equiv 0$

We thus obtain for $p\equiv 3 \pmod 8$ two types for which $J_A=(P_1, P_2, P_3; P_1^{\pm 1}, P_3, P_2P_3^c)$

(*ii*) $b \equiv -1$ $\quad \because c^2 \equiv 2, \quad a \equiv \pm 1, \quad a \equiv 0$

We thus obtain for $p \equiv 7 \pmod 8$ two types for which $J_A = (P_1, P_2, P_3;\ P_1^{\pm 1}, P_3, P_2^{-1}P_3^{c})$.

3. The permutation contains cycles with only eight terms. When P_2 is transformed by A^4 we get, e. g., P_3 or $P_1^{x}P_2^{y}$. In both cases we are brought back to the previous case. P_2 is exchanged for P_2P_3 or a suitable operation in $\{P_1, P_2\}$

$$G_{8p^3}/G_{p^3}^3 = \{A^4 = B^2 = 1 \qquad AB = BA\}$$

$H = \{A^2, B\}$ or $\{A\}$. Six types.

$H = \{B\}$. We get nine types for $p \equiv 1 \pmod 4$ and two for $p \equiv 3 \pmod 4$.

$H = \{A^2\}$.

In $G_{4p^3} = \{A, P_1, P_2, P_3\}$ J_A is an isomorphism of order 2. We can suppose $J_A = (P_1, P_2, P_3;\ P_1^{\alpha}, P_2^{\beta}, P_3^{\gamma})$ where α, β and γ are roots of $\delta^2 \equiv 1 \pmod p$. J_B, which is a general isomorphism of order 2, must be permutable with J_A.

We have

$$J_B = (P_1, P_2, P_3;\quad P_1^{x_1}P_2^{y_1}P_3^{z_1},\quad P_1^{x_2}P_2^{y_2}P_3^{z_2},\quad P_1^{x_3}P_2^{y_3}P_3^{z_3})$$

The isomorphisms J_A and J_B are permutable if

$$\left.\begin{aligned} y_1(\alpha-\beta) \equiv x_2(\alpha-\beta) \equiv x_3(\alpha-\gamma) \equiv 0 \\ z_1(\alpha-\gamma) \equiv z_2(\beta-\gamma) \equiv y_3(\beta-\gamma) \equiv 0 \end{aligned}\right\} \pmod p \quad . \quad . \quad . \quad (2)$$

1. $\alpha \equiv \beta \equiv \gamma \equiv -1$

Every G_p in $G_{p^3}^3$ is transformed into itself by the isomorphism J_A, and since for J_B the number of self-conjugate G_p is ≥ 3, we can take $J_B = (P_1, P_2, P_3;\ P_1^{x_1}, P_2^{y_2}, P_3^{z_3})$, where x_1, y_2 and z_3 are roots of $\delta^2 \equiv 1$. We obtain two types for which $J_B = (P_1, P_2, P_3;\ P_1^{-1}, P_2^{\pm 1}, P_3)$.

2. $\alpha \equiv \beta \equiv -1,\ \gamma \equiv 1$

$$\because z_1 \equiv z_2 \equiv x_3 \equiv y_3 \equiv 0$$

Instead of P_1 and P_2 we choose new generating operations, so that $J_B = (P_1, P_2, P_3; P_1^{x_1}, P_2^{y_2}, P_3^{z_3})$. We obtain three new types for which $J_B = (P_1, P_2, P_3; P_1^{-1}, P_2^{\pm 1}, P_3^{-1})$ or $(P_1, P_2, P_3; P_1^{-1}, P_2, P_3)$

3. $\alpha \equiv -1, \ \beta \equiv \gamma \equiv 1$

$$\because \ y_1 \equiv z_1 \equiv x_2 \equiv x_3 \equiv 0$$

Instead of P_2 and P_3 we choose new generating operations, so that we can suppose $J_B = (P_1, P_2, P_3; P_1^{x_1}, P_2^{y_2}, P_3^{z_3})$. We obtain only one new type for which ($J_B = (P_1, P_2, P_3; P_1^{-1}, P_2, P_3^{-1})$.

$H = E$

In $G_{4p^3} = \{A, P_1, P_2, P_3\}$ J_A is an isomorphism of order 4. This isomorphism can be chosen in nine different ways for $p \equiv 1 \pmod 4$ and in two ways for $p \equiv 3 \pmod 4$. J_B is an isomorphism of order 2 permutable with J_A.

(*i*) $J_A = (P_1, P_2, P_3; P_1^{\alpha}, P_2^{\beta}, P_3^{\gamma})$ where α, β and γ roots of $\delta^4 \equiv 1$. The isomorphisms J_A and J_B are permutable if congruences (2) are satisfied.

1. $\alpha \equiv \beta \equiv \gamma \equiv \delta$

Every G_p in $G_{p^3}^3$ is transformed into itself by the isomorphism J_A. Since J_B is of order 2 the number of self-conjugate G_p is ≥ 3 and we can thus choose $J_B = (P_1, P_2, P_3; P_1^{x_1}, P_2^{y_2}, P_3^{z_3})$ where x_1, y_2 and z_3 are roots of $\delta^2 \equiv 1$. We obtain a single type for which $J_B = (P_1, P_2, P_3; P_1^{-1}, P_2^{-1}, P_3)$.

2. $\alpha \equiv \beta \equiv \delta, \ \gamma \equiv \delta^3$ or ± 1

$$\because \ z_1 \equiv z_2 \equiv x_3 \equiv y_3 \equiv 0.$$

By choosing new generating operations of $\{P_1, P_2\}$ we can suppose $J_B = (P_1, P_2, P_3; P_1^{x_1}, P_2^{y_2}, P_3^{z_3})$. We obtain six new types for which

$$J_A = (P_1,\ P_2,\ P_3;\ P_1^{\delta},\ P_2^{\delta},\ P_3^{\delta^2}) \quad J_B = (P_1,\ P_2,\ P_3;\ P_1^{-1},\ P_2,\ P_1^{-1})$$

$$J_A = (P_1,\ P_2,\ P_3;\ P_1^{\delta},\ P_2^{\delta},\ P_3) \quad J_B = (P_1,\ P_2,\ P_3;\ P_1^{-1},\ P_2^{\pm 1},\ P_3^{-1})$$

$$J_A = (P_1,\ P_2,\ P_3;\ P_1^{\delta},\ P_2^{\delta},\ P_3) \quad J_B = (P_1,\ P_2,\ P_3;\ P_1,\ P_2^{-1},\ P_3)$$

$$J_A = (P_1,\ P_2,\ P_3;\ P_1^{\delta},\ P_2^{\delta},\ P_3^{-1}) \quad J_B = (P_1,\ P_2,\ P_3;\ P_1^{-1},\ P_2,\ P_3^{-1})$$

or $J_A = (P_1,\ P_2,\ P_3;\ P_1^{\delta},\ P_2^{\delta},\ P_3^{-1}) \quad J_B = (P_1,\ P_2,\ P_3;\ P_1,\ P_2^{-1},\ P_3)$

3. $\alpha \equiv \delta,\ \beta \equiv \delta^3,\ \gamma \equiv \pm 1$

$$\because\ x_1 \equiv y_1 \equiv z_1 \equiv \ldots\ldots \equiv 0$$

No isomorphism J_B exists

4. $\alpha \equiv \delta,\ \beta \equiv \gamma \equiv \pm 1$

$$\because\ y_1 \equiv z_1 \equiv x_2 \equiv x_3 \equiv 0$$

By choosing new generating operations of $\{P_2,\ P_3\}$ we can suppose $J_B = (P_1,\ P_2,\ P_3;\ P_1^{x_1},\ P_2^{y_2},\ P_3^{z_3})$. We obtain three types for which

$$J_A = (P_1, P_2, P_3; P_1^{\delta}, P_2, P_3) \quad J_B = (P_1, P_2, P_3; P_1^{-1}, P_2^{-1}, P_3^{\pm 1})$$

or $J_A = (P_1, P_2, P_3; P_1^{\delta}, P_2^{-1}, P_3^{-1}) \quad J_B = (P_1,\ P_2,\ P_3;\ P_1^{-1},\ P_2^{-1},\ P_3)$

5. $\alpha \equiv \delta,\ \beta \equiv -1,\ \gamma \equiv +1$

$$\because\ x_1 \equiv y_1 \equiv z_1 \equiv \ldots\ldots \equiv 0$$

No isomorphism J_B exists.

(*ii*) $J_A = (P_1,\ P_2,\ P_3;\ P_1^{a},\ P_3,\ P_2^{-1})$, where $a^2 \equiv 1$. The isomorphisms J_A and J_B are permutable if

$$\left.\begin{array}{l} ay_1 + z_1 \equiv ax_2 - x_3 \equiv y_3 + z_2 \equiv 0 \\ az_1 - y_1 \equiv ax_3 + x_2 \equiv y_2 - z_3 \equiv 0 \end{array}\right\} \pmod{p}$$

$$\because\ y_1 \equiv z_1 \equiv x_2 \equiv x_3 \equiv 0$$

Since J_B is an isomorphism of order 2 we get

$$x_1^2 \equiv 1, \quad y_2^2 - z_2^2 \equiv 1, \quad 2y_2z_2 \equiv 0$$

Hence it follows that

$$z_2 \equiv 0, \quad y_2 \equiv z_3 \equiv \pm 1$$

We obtain for $p \equiv 3 \pmod 4$ a new type for which

$J_A = (P_1, P_2, P_3;\ P_1, P_3, P_2^{-1}) \quad J_B = (P_1, P_2, P_3;\ P_1^{-1}, P_2, P_3)$

$G_{8p^3}/G_{p^3}^3 = \{A^2 = B^2 = C^2 = 1 \quad AB = BA \quad AC = CA \quad BC = CB\}$

$H = \{A, B\}$. Three types.

$H = \{C\}$.

We obtain three types for which

$J_A = (P_1, P_2, P_3;\ P_1^{-1}, P_2^{-1}, P_3^{\pm 1}) \ J_B = (P_1, P_2, P_3;\ P_1, P_2^{-1}, P_3^{-1})$

or $J_A = (P_1, P_2\, P_3;\ P_1^{-1}, P_2, P_3) \ J_B = (P_1, P_2, P_3;\ P_1^{-1}, P_2^{-1}, P_3)$

$H = E$

In $G_{4p^3} = \{A, B, P_1, P_2, P_3\}$ we can choose J_A and J_B, as was previously shown. J_C is permutable with J_A and J_B.

$$\because \quad J_C = (P_1, P_2, P_3;\ P_1^x, P_2^y, P_3^z)$$

We get a single type for which the isomorphisms are $(P_1, P_2, P_3;\ P_1^{\pm 1}, P_2^{\pm 1}, P_3^{\pm 1})$

$G_{8p^3}/G_{p^3}^3 = \{A^4 = B^2 = 1 \quad B^{-1}AB = A^3\}$

$H = \{A\}$ or $\{A^2, B\}$. Six types.

$H = \{A^2\}$.

Six types for which

$J_A = (P_1, P_2, P_3;\ P_1^{-1}, P_2^{-1}, P_3^{-1}) \ J_B = (P_1, P_2, P_3;\ P_1, P_2, P_3^{-1})$

$J_A = (P_1, P_2, P_3;\ P_1^{-1}, P_2^{-1}, P_3) \ J_B = (P_1, P_2, P_3;\ P_1^{-1}, P_2^{-1}, P_3^{-1})$

$J_A = (P_1, P_2, P_3;\ P_1^{-1}, P_2^{-1}, P_3) \ J_B = (P_1, P_2, P_3;\ P_1^{-1}, P_2, P_3^{\pm 1})$

or $J_A = (P_1, P_2, P_3;\ P_1^{-1}, P_2, P_3) \ J_B = (P_1, P_2, P_3;\ P_1^{-1}, P_2^{\pm 1}, P_3^{-1})$

$H = E$

The isomorphisms corresponding to A and B must satisfy

$$J_A J_B = J_B J_{A^3}$$

As before it is shown that we can suppose

(*i*) $J_A = (P_1, P_2, P_3;\ P_1^{\alpha}, P_2^{\beta}, P_3^{\gamma})$

$$\because \left.\begin{aligned} x_1\alpha(\alpha^2-1) \equiv y_1(\beta^3-\alpha) \equiv z_1(\gamma^3-\alpha) \equiv 0 \\ x_2(\alpha^3-\beta) \equiv y_2\beta(\beta^2-1) \equiv z_2(\gamma^3-\beta) \equiv 0 \\ x_3(\alpha^3-\gamma) \equiv y_3(\beta^3-\gamma) \equiv z_3\gamma(\gamma^2-1) \equiv 0 \end{aligned}\right\} \pmod{p}$$

$$\because\ \alpha \equiv \delta, \quad \beta \equiv \delta^3, \quad \gamma \equiv \pm 1$$

$$\because\ x_1 \equiv x_3 \equiv y_2 \equiv y_3 \equiv z_1 \equiv z_2 \equiv 0$$

Since J_B is an operation of order 2 we get

$$z_3^2 \equiv 1, \quad x_2y_1 \equiv 1$$

When $p \equiv 1 \pmod 4$, we thus get three types, for which

$J_A = (P_1, P_2, P_3;\ P_1^{\delta}, P_2^{\delta^3}, P_3)$ $J_B = (P_1, P_2, P_3;\ P_2, P_1, P_3^{\pm 1})$
or $J_A = (P_1, P_2, P_3;\ P_1^{\delta}, P_2^{\delta^3}, P_3^{-1})$ $J_B = (P_1, P_2, P_3;\ P_2, P_1, P_3)$

(*ii*) $J_A = (P_1, P_2, P_3;\ P_1^{\alpha}, P_3, P_2^{-1})$

$$\because \left.\begin{aligned} x_1\alpha(\alpha^2-1) \equiv \alpha z_1 + y_1 \equiv \alpha y_1 - z_1 \equiv 0 \\ y_2 + z_3 \equiv y_3 - z_2 \equiv \alpha^3 x_2 - x_3 \equiv 0 \\ \alpha^3 x_3 + x_2 \equiv 0 \end{aligned}\right\} \pmod{p}$$

$$\because\ y_1 \equiv z_1 \equiv x_2 \equiv x_3 \equiv 0$$

Since J_B is an operation of order 2 we get

$$x_1^2 \equiv 1, \quad y_2^2 + z_2^2 \equiv 1$$

When $p \equiv 3 \pmod 4$ we thus get three types for which

$J_A = (P_1, P_2, P_3;\ P_1, P_3, P_2^{-1})$ $J_B = (P_1, P_2, P_3;\ P_1^{\pm 1}, P_2, P_3^{-1})$
or $J_A = (P_1, P_2, P_3;\ P_1^{-1}, P_3, P_2^{-1})$ $J_B = (P_1, P_2, P_3;\ P_1, P_2, P_3^{-1})$

$G_{8p^3}/G_{p^3}^3 = \{A^4 = B^4 = 1 \quad A^2 = B^2 \quad B^{-1}AB = A^3\}$

$H = \{A\}$. Three types.

$H = \{A^2\}$.

We obtain three types for which

$J_A = (P_1, P_2, P_3;\ P_1^{-1}, P_2^{-1}, P_3^{-1})$ $J_B = (P_1, P_2, P_3;\ P_1, P_2, P_3^{-1})$

or $J_A = (P_1, P_2, P_3;\ P_1^{-1}, P_2^{-1}, P_3)$ $J_B = (P_1, P_2, P_3;\ P_1, P_2^{-1}, P_3^{\pm 1})$

$H = E$

The isomorphisms corresponding to A and B must satisfy

$$J_A J_B = J_B J_{A^3}$$

$$J_{A^2} = J_{B^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

As before it is shown that we can suppose

(*i*) $J_A = (P_1, P_2, P_3; P_1^\alpha, P_2^\beta, P_3^\gamma)$

$$\because \alpha \equiv \delta, \quad \beta \equiv \delta^3, \quad \gamma \equiv \pm 1$$

$$\because x_1 \equiv x_3 \equiv y_2 \equiv y_3 \equiv z_1 \equiv z_2 \equiv 0$$

From the relation (3) we get

$$z_3^2 \equiv 1, \quad y_1 x_2 \equiv -1$$

When $p \equiv 1 \pmod 4$ we thus get two types for which

$J_A = (P_1, P_2, P_3; P_1^\delta, P_2^{\delta^3}, P_3) \quad J_B = (P_1, P_2, P_3; P_2^{-1} P_1, P_3^{\pm 1})$

(*ii*) $J_A = (P_1, P_2, P_3; P_1^\alpha, P_3, P_2^{-1})$

$$\because y_1 \equiv z_1 \equiv x_2 \equiv x_3 \equiv 0$$

From the relation (3) we get

$$x_1^2 \equiv 1, \quad y_2^2 + z_2^2 \equiv -1$$

When $p \equiv 3 \pmod 4$ we thus get two types for which

$J_A = (P_1, P_2, P_3; P_1, P_3, P_2^{-1})$ $J_B = (P_1, P_2, P_3; P_1^{\pm 1}, P_2^{y_2} P_3^{z_2}, P_2^{z_2} P_3^{-y_2})$ where z_2, y_2 is a fixed solution of $y_2^2 + z_2^2 \equiv -1$.

$$G_{p^3}^6 = \{P_1^{p^2} = P_2^p = 1 \quad P_2^{-1} P_1 P_2 = P_1^{1+p}\}$$

$G_{8p^3}/G_{p^3}^6 = \{A^8 = 1\}$

A must be permutable with at least one of the sub-groups $\{P_1 P_2^k\}$ $(k = 0, 1 \,.\,.\, p-1)$. For this we can choose $\{P_1\}$. In addition to $\{P_1^p\}$ A is also permutable with one of the sub-groups $\{P_1^{kp} P_2\}$ $(k = 0, 1 \,.\,.\, p-1)$, for instance $\{P_2\}$

$$\because A^{-1} P_1 A = P_1^\alpha \qquad A^{-1} P_2 A = P_2^\beta$$

But $$P_2^{-\beta} P_1^{\alpha} P_2^{\beta} = P_1^{\alpha(1+\beta p)}$$

$$\because \ \beta \equiv 1 \pmod{p}$$

$H = \{A^2\}$ $\quad \because \ J_A = (P_1, \ P_2; \ P_1^{-1}, \ P_2)$

$H = \{A^4\}$ $\quad \because \ J_A = (P_1, \ P_2; \ P_1^{\alpha}, \ P_2)$. α a fixed prim. root of $\alpha^4 \equiv 1$

$H = E$ $\quad \because \ J_A = (P_1, \ P_2; \ P_1^{\alpha}, \ P_2)$. α a fixed prim. root of $\alpha^8 \equiv 1$.

The group of isomorphisms of $G_{p^3}^6$ is of order $p^3(p-1)$. The conjugated sequence of Sylow's sub-groups of order 2^k to this group consists, as can be easily seen, of cyclical $G_2{}^k$. Groups in this category thus only exist when $G_8/_H$ is isomorphic with a cyclical group. Six new types are obtained for each value of p. When $p \equiv 1 \pmod 4$ there is one more, as in that case the factor-group $G_8/_H$ can be isomorphic with a cyclic G_4.

$$G_{p^3}^7 = \left\{ \begin{matrix} P_1^p = P_2^p = P_3^p = 1 & P_1 P_2 = P_2 P_1 & P_1 P_3 = P_3 P_1 \\ & P_3^{-1} P_2 P_3 = P_1 P_2 & \end{matrix} \right\}$$

$G_{8p^3}/G_{p^3}^7 = \{A^8 = 1\}$

P_1 is a characteristic sub-group of $G_{p^3}^7$. This is self-conjugate when transformed by A. Of the remaining $p^2 + p$ sub-groups of order p either none or at least two are permutable with A. In the latter case we can suppose

$$A^{-1} P_1 A = P_1^{\alpha} \quad A^{-1} P_2 A = P_2^{\beta}$$

Independently of whether $\alpha \equiv \beta \pmod{p}$ or not we can choose as the third group $\{P_3\}$

$$\because \ J_A = (P_1, \ P_2, \ P_3; \ P_1^{\alpha}, \ P_2^{\beta}, \ P_3^{\gamma})$$

$$\because \ \alpha^8 \equiv \beta^8 \equiv \gamma^8 = 1 \quad \alpha \equiv \beta\gamma \pmod{p}$$

$H = \{A^2\}$

$\because \ J_A = (P_1, P_2, P_3; P_1, P_2^{-1}, P_3^{-1})$ or $(P_1 \, P_2, P_3; P_1^{-1}, P_2 \, P_3^{-1})$

$H = \{A^4\}$ $\because$ $J_A = (P_1, P_2, P_3;\ P_1^{a^r+1}, P_2^{a^r}, P_3^{a})$ $(r = 0, 1, 2, 3)$

$H = E$ $\because$ $J_A = (P_1, P_2, P_3;\ P_1^{a^r+1}, P_2^{a^r}, P_3^{a})$ $(r \neq 1, 5)$

When A is only permutable with $\{P_1\}$, the other p^2+p G_p are permuted in cyclical sequences with 2, 4 or 8 groups in each.

We may thus assume that

$$J_A = (P_1, P_2, P_3;\ P_1^{a}, P_3, P_1^{a}P_2^{b}P_3^{c})$$
$$\because\ a \equiv -b$$

1. The permutation contains one cycle with two termes

$$\because\ a = c \equiv 0 \quad b^4 = 1 \pmod{p}$$

Supposing $b \equiv 1$ J_A is an isomorphism for which more than one G_p is self-conjugate. If $b \equiv -1$, we obtain for $p \equiv 3 \pmod 4$ one type for which $J_A = (P_1, P_2, P_3;\ P_1, P_3, P_2^{-1}$. If on the other hand b is a prim. root, we obtain for $p \equiv 5 \pmod 8$ one type for which $J_A = (P_1, P_2, P_3;\ P_1^{b^3}, P_3, P_2^{b})$.

2. The permutation contains cycles with at least four terms.

$$J_A = (P_1, P_2, P_3;\ P_1^{a}, P_3, P_1^{a}P_2^{b}P_3^{c})$$
$$J_{A^2} = (P_1, P_2, P_3;\ P_1^{a^2}, P_1^{a}P_2^{b}P_3^{c}, P_1^{x}P_2^{bc}P_3^{b+c^2})$$
$$J_{A^4} = (P_1, P_2, P_3;\ P_1^{a^4}, P_1^{y}P_2^{b(b+c^2)}P_3^{2bc+c^2}, \ldots .)$$

where $x = aa + ac - \frac{1}{2}bc^2(c-1) - b^2c$

$$y = xa + a(b+c^2) - \frac{1}{2}bc(b+c^2)\ (b+c^2-1) - b^2c(b+c^2)$$

$$\because \left.\begin{array}{r} y \equiv 0 \\ b^2 + bc^2 \equiv \pm 1 \\ 2bc + c^3 \equiv 0 \end{array}\right\} \pmod{p}$$

Because $p^2 + p = 4k$, it is necessary that $p \equiv 3 \pmod 4$. When $c \equiv 0$, the permutation corresponding to J_A contains one cycle with two terms

$$\because\ 2b + c^2 \equiv 0$$

Hence it follows that

$$b^2 \equiv 1$$

(i) $b \equiv 1 \qquad \because a \equiv -1, \quad c^2 \equiv -2, \quad ac = 1$

A fixed solution of this congruences give for $p \equiv 3$ (mod 8) a type for which $J_A = (P_1, P_2, P_3;\ P_1^{-1}, P_3, P_1^a P_2 P_3^c)$. The solution $-c$ and $-a$ give a group isomorphic with the foregoing. To show this we may, e. g., let $\{A, P_1, P_3, P_3\}$ be generated by $\{A^5, P_1^{-1}, P_2, P_3^{-1}\}$.

(ii) $b = -1 \quad \because a \equiv 1, \quad c^2 \equiv 2, \quad a(c+2) - c - 1 \equiv 0$

We obtain for $p \equiv 7$ (mod 8) a single type for which $J_A = (P_1, P_2, P_3;\ P_1, P_3, P_1^a P_2^{-1} P_3^{2a})$.

3. The permutation contains cycles with only eight terms. When P_2 is transformed by A^4 we get P_3 or $P_1^x P_2^y$. In both cases we are brought back to the previous case. P_2 is exchanged for $P_2 P_3$ or a suitable operation in $\{P_1, P_3\}$.

$G_{8p^3}/G_{p^3}^7 = \{A^4 = B^2 = 1 \quad AB = BA\}$.

$H = \{A^2, B\}$ or $\{A\}$. Four types.

$H = \{B\}$

We obtain four types for $p \equiv 1$ (mod 4) and a single one for $p \equiv 3$ (mod 4)

$H = \{A^2\}$

We may take

$J_A = (P_1, P_2, P_3;\ P_1^\alpha, P_2^\beta, P_3^\gamma)$. α, β and γ roots of $\delta^2 \equiv 1$

$J_B = (P_1, P_2, P_3;\ P_1^x, P_1^{x_1} P_2^{y_1} P_3^{z_1}, P_1^{x_2} P_2^{y_2} P_3^{z_2})$ is an isomorphism of $G_{p^3}^7$ if

$$x \equiv y_1 z_2 - y_2 z_1$$

From $J_{B^2} = 1$ we get

$$x^2 \equiv 1 \quad \ldots \quad (4)$$

$$x x_1 + x_1 y_1 - \frac{1}{2} y_1^2 z_1 (y_1 - 1) + x_2 z_1 - \frac{1}{2} y_2 z_1 z_2 (z_1 - 1) - y_1 y_2 z_1^2 \equiv 0 \quad \ldots \quad (5)$$

$$y_1^2 + y_2 z_1 \equiv 1 \quad z_1 (y_1 + z_2) \equiv 0 \quad \ldots \quad (6)$$

$$xx_2 + x_1y_2 - \frac{1}{2}y_1y_2z_1(y_2-1) + x_2z_2 - \frac{1}{2}y_2z_2^2(z_2 - 1) - y_2^2z_1z_2 \equiv 0 \quad \ldots (7)$$

$$z_2^2 + y_2z_1 \equiv 1 \quad y_2(y_1 + z_2) \equiv 0 \ldots (8)$$

The isomorphisms J_A and J_B are permutable if

$$x_1(\beta - \alpha) - \frac{1}{2}y_1z_1\beta(\beta - 1) \equiv 0$$

$$x_2(\gamma - \alpha) - \frac{1}{2}y_2z_2\gamma(\gamma - 1) \equiv 0$$

$$z_1(\beta - \gamma) \equiv y_2(\beta - \gamma) \equiv 0$$

(*i*) $\alpha \equiv \gamma \equiv -1, \quad \beta \equiv 1$

$$\because \; x_1 \equiv z_1 \equiv y_2 \equiv 0, \quad x \equiv y_1z_2$$

From the relations (4—8) we get

$$x^2 \equiv y_1^2 \equiv z_2^2 \equiv 1 \qquad x_2(x + z_2) \equiv 0$$

By varying the generating operations of $\{P_1, P_3\}$ we show that only two types are distinct viz. those for which

$$J_A = (P_1, P_2, P_3; P_1^{-1}, P_2, P_3^{-1}) \; J_B = (P_1, P_2, P_3; P_1, P_2^{-1}, P_3^{-1})$$

or $J_A = (P_1, P_2, P_3; P_1^{-1}, P_2, P_3^{-1}) \; J_B = (P_1, P_2, P_3; P_1^{-1}, P_2^{-1}, P_3)$

(*ii*) $\alpha \equiv 1 \quad \beta \equiv \gamma \equiv -1$

$$\because \; 2x_1 + y_1z_1 \equiv 2x_2 + y_2z_2 \equiv 0 \quad \ldots\ldots (9)$$

From (6, 8) we get

$$y_1^2 \equiv z_2^2$$

1. $y_1 \equiv z_2 \not\equiv 0 \qquad \because \; z_1 \equiv y_2 \equiv 0$

Hence it follows that

$$x \equiv 1 \quad x_1 \equiv x_2 \equiv 0$$

J_B is in this case identical with J_A.

2. $y_1 \equiv -z_2 \qquad \because \; y_1^2 + z_1y_2 \equiv 1 \qquad x \equiv -1$

When x_1 and x_2 are determined by (9) the relations (5, 7) are also satisfied. In order to show that the solutions give isomorphic groups we wary the generating operations of $G_{p^3}^7$.

We may suppose

$$O_1 = P_1^a, \quad O_2 = P_1^{a_1} P_2^{b_1} P_3^{c_1}, \quad O_3 = P_1^{a_2} P_2^{b_2} P_3^{c_2}$$

$$\because \ a \equiv b_1 c_2 - b_2 c_1$$

J_A preserves the same type, if

$$2a_1 + b_1 c_1 \equiv 0 \quad 2a_2 + b_2 c_2 \equiv 0.$$

The values $y_1 \equiv 1 \quad z_2 \equiv -1 \quad x_1 \equiv x_2 \equiv z_1 \equiv y_2 \equiv 0$ give a type and we can always determine a, a_1, b_1, . . . so that

$$J_B = (O_1, O_2, O_3;\ O_1^{-1},\ O_1^{x_1} O_2^{y_1} O_3^{z_1},\ O_1^{x_2} O_2^{y_2} O_3^{z_2})$$

where x_1, y_1, . . . is a fixed solution of the congruences. The group obtained is isomorphic with the previous one.

$H = E$

(*i*) $J_A = (P_1, P_2, P_3;\ P_1^{a^r+1},\ P_2^{a^r},\ P_3^{a})$ is permutable with J_B if

$$x_1 a^r (1 - a) - \frac{1}{2} y_1 z_1 a^r (a^r - 1) \equiv 0$$

$$x_2 a (1 - a^r) - \frac{1}{2} y_2 z_2 a (a - 1) \equiv 0$$

$$z_1 a (a^{r-1} - 1) \equiv y_2 a (a^{r-1} - 1) \equiv 0$$

1. $r \equiv 0 \qquad \because \ z_1 \equiv y_2 \equiv x_1 \equiv x_2 \equiv 0$

From (4—8) we get

$$y_1^2 \equiv z_2^2 \equiv 1$$

We obtain a single type for which

$$J_A = (P_1, P_2, P_3;\ P_1^a, P_2, P_3^a) \quad J_B = (P_1, P_2, P_3;\ P_1, P_2^{-1}, P_3^{-1})$$

2. $r \equiv 1 \qquad \because \ 2x_1 + y_1 z_1 \equiv 0 \quad 2x_2 + y_2 z_2 \equiv 0$

From (4—8) we get

$$y_1 \equiv -z_2 \qquad y_1^2 + z_1 y_2 \equiv 1 \qquad x \equiv -1$$

We obtain a single type for which

$$J_A = (P_1, P_2, P_3;\ P_1^{-1}, P_2^a, P_3^a) \quad J_B = (P_1, P_2, P_3;\ P_1^{-1}, P_2, P_3^{-1})$$

3. $r \equiv 2,\ 3$ $\qquad \because\ z_1 \equiv y_2 \equiv x_1 \equiv x_2 \equiv 0$

From (4—8) we get

$$y_1^2 \equiv z_2^2 \equiv 1$$

The groups are isomorphic with foregoing.

(*ii*) $J_A = (P_1,\ P_2,\ P_3;\ P_1,\ P_3,\ P_2^{-1})$ is permutable with J_B

if $\qquad y_2 + z_1 \equiv y_1 - z_2 \equiv x_2 - x_1 - y_1 z_1 \equiv 2x_2 + y_2 z_2 \equiv 0$

The relations (4—8) give in combination with these

$z_1 \equiv x_1 \equiv x_2 \equiv y_2 \equiv 0 \quad y_1^2 \equiv 1,$ which gives no group belonging here.

$G_{8p^3}/G^7_{p^3} = \{A^2 = B^2 = C^2 = 1 \quad AB = BA \quad AC = CA \quad BC = CB\}$

$H = \{A,\ B\}$. Two types.

$H = \{C\}$

$\because\ J_A = (P_1,\ P_2,\ P_3;\ P_1,\ P_2^{-1},\ P_3^{-1})\ J_B = (P_1, P_2, P_3; P_1^{-1}, P_2, P_3^{-1})$

$H = E$

J_A and J_B can be chosen as was shown before. J_C is permutable with both and is thus an isomorphism in $\{J_A,\ J_B\}$. No group exists

$G_{8p^3}/G^7_{p^3} = \{A^4 = B^2 = 1 \quad B^{-1}AB = A^3\}$

$H = \{A\}$ or $\{A^2,\ B\}$. Four types.

$H = \{A^2\}$

We obtain two types for which

$J_A = (P_1, P_2, P_3;\ P_1^{-1},\ P_2, P_3^{-1}) \quad J_B = (P_1, P_2 P_3;\ P_1, P_2^{-1}, P_3^{-1})$

or $J_A = (P_1, P_2, P_3;\ P_1, P_2^{-1}, P_3^{-1}) \quad J_B = (P_1, P_2 P_3;\ P_1^{-1}, P_2, P_3^{-1})$

$H = E$

The isomorphisms J_A and J_B satisfy the relation

$$J_A J_B = J_B J_{A^3}$$

(i) $J_A = (P_1,\ P_2,\ P_3;\ P_1^{a^r+1},\ P_2^{a^r},\ P_3^{a})$

$$\because\ x_1(a^r - a^{3(r+1)}) - \frac{1}{2}y_1z_1a^r(a^r-1) \equiv 0$$

$$x_2(a - a^{3(r+1)}) - \frac{1}{2}y_2z_2a(a-1) \equiv 0$$

$$y_1(a^{3r} - a^r) \equiv z_1(a^3 - a^r) \equiv 0$$

$$y_2(a^{3r} - a) \equiv z_2(a^3 - a) \equiv 0$$

$$x(a^{3(r+1)} - a^{r+1}) \equiv 0$$

$$\because\ r \equiv 3, \quad x_1 \equiv x_2 \equiv y_1 \equiv z_2 \equiv 0$$

From (4—8) we get

$$x \equiv -1, \quad y_2z_1 \equiv 1$$

We get for $r \equiv 1 \pmod 4$ a single type for which

$J_A = (P_1,\ P_2,\ P_3;\ P_1,\ P_2^{a^3},\ P_3^{a})\quad J_B = (P_1,\ P_2\ P_3;\ P_1^{-1},\ P_3,\ P_2)$

(ii) $J_A = (P_1,\ P_2,\ P_3;\ P_1,\ P_3,\ P_2^{-1})$

$$\because\ y_2 - z_1 \equiv y_1 + z_2 \equiv x_2 - x_1 - y_1z_1 \equiv 2x_2 + y_2z_2 \equiv 0$$

From (4—8) we get

$$x \equiv -1, \quad y_1^2 + z_1^2 \equiv 1$$

We obtain for $p \equiv 3 \pmod 4$ a single type for which

$J_A = (P_1,\ P_2,\ P_3;\ P_1,\ P_3,\ P_2^{-1})\quad J_B = (P_1,\ P_2,\ P_3;\ P_1^{-1},\ P_2,\ P_3^{-1})$

$G_{8p^3}/G_{p^3}^{7} = \{A^4 = B^4 = 1 \quad A^2 = B^2 \quad B^{-1}AB = A^3\}$

$H = \{A\}$. Two types.

$H = \{A^2\}$.

$\because\ J_A = (P_1,\ P_2,\ P_3;\ P_1^{-1},\ P_2,\ P_3^{-1})\ J_B = (P_1,\ P_2,\ P_3;\ P_1,\ P_2^{-1},\ P_3^{-1})$

$H = E$

The isomorphisms J_A and J_B of order 4 satisfy the relations

$$J_AJ_B = J_BJ_{A^3}$$

$$J_{A^2} = J_{B^2} \quad \ldots\ldots\ldots \quad (10)$$

(*i*) $J_A = (P_1, P_2, P_3; P_1^{a^r+1}, P_2^{a^r}, P_3^{a})$

$$\because r \equiv 3, \quad x_1 \equiv x_2 \equiv y_1 \equiv z_2 \equiv 0$$

From (10) we get

$$x \equiv 1, \quad y_2 z_1 \equiv -1$$

We obtain for $p \equiv 1 \pmod 4$ a single type for which

$$J_A = (P_1, P_2, P_3; P_1, P_2^{a^3}, P_3^{a}) \quad J_B = (P_1, P_2, P_3; P_1, P_3^{-1}, P_2)$$

(*ii*) $J_A = (P_1, P_2, P_3; P_1, P_3, P_2^{-1})$

$$\because y_2 - z_1 \equiv y_1 + z_2 \equiv 2x_1 + y_1 z_1 \equiv 2x_2 + y_2 z_2 \equiv 0$$

From (10) we get

$$x \equiv 1, \quad y_1^2 + z_1^2 \equiv -1$$

We obtain for $p \equiv 3 \pmod 4$ a single type for which

$$J_A = (P_1, P_2, P_3; P_1, P_3, P_2^{-1})$$
$$J_B = (P_1, P_2, P_3; P_1, P_1^{x_1} P_2^{y_1} P_3^{z_1}, P_1^{-x_1} P_2^{z_1} P_3^{-y_1})$$

where x_1, y_1 and z_1 are a fixed solution of the congruences.

III.

The G_{8p^3} which have a self-conjugate G_8 and more than one G_{p^3}.

If the operations in G_8 are transformed with G_{p^3} we obtain the same operations in another order. Every operation in G_{p^3} thus corresponds to an isomorphism of G_8. Since the group of isomorphisms of G_8^r $(r = 3, 5)$ is only divisible by p, every operation in a sub-group H of order p^2 of G_{p^3} must be permutable with every operation in G_8^r. The groups are thus obtained direct from Western's treatise[1].

[1] Western, Groups of order $p^3 q$. L. M. S. Proc. Vol. 30 (1899).

We have the following cases to consider:

$G^1_{p^3}$. $H=\{P^p\}$. We obtain three types.

$\{G^3_8, G^1_{27}\}$ for which $J_P=(A)\ (B\ C\ BC)\ .\ .\ .$

$\{G^5_8, G^1_{27}\}$ for which $J_P=(A^2)\ (A\ B\ AB)\ .\ .\ .$

$\{G^3_8, G^1_{7^3}\}$ for which $J_P=(A\ B\ C\ AB\ .\ .\ .)$

$G^2_{p^3}$. $H=\{P_1\}$ or $\{P_1{}^p, P_2\}$. Six types.

$G^3_{p^3}$. $H=\{P_1, P_2\}$. Three types.

$G^6_{p^3}$. $H=\{P_1\}$ or $\{P_1{}^p, P_2\}$. Six types.

$G^7_{p^3}$. $H=\{P_1, P_2\}$. Three types.

IV.

The G_{8p^3} which contain no self-conjugate G_8 or G_{p^3}.

(*i*) $p=7$.

All the groups $G_{8.7^3}$ contain 8 G_{7^3}. These 8 G_{7^3} have a common sub-group G_{49}, which is self-conjugate in $G_{8.7^3}$. The factor-group $G_{8.7^3}/G_{49}=\Gamma_{56}$ which is formed by this has no self-conjugate sub-group of order 7. In such a case $G_{8.7^3}$ would have a self-conjugate G_{7^3} which is contrary to the hypothesis. Γ_{56} thus has a self-conjugate G_8, which is necessarily an Abelian group of type (1, 1, 1). $G_{8.7^3}$ has thus a self-conjugate $G_{8.7^2}$. This har 7 or 49 G_8, and since the group of isomorphisms of G_{p^2} contains no Abelian G_8 of type (1, 1, 1), these G_8 have a common sub-group H of order 2 or 4. H is self-conjugate in $G_{8.7^3}$. The factor-groups $G_{8.7^3}/H=\Gamma_{2.7^3}$ or $\Gamma_{4.7^3}$ which are formed by this have a self-conjugate sub-group G_{7^3} (Sylow's theorem). $G_{8.7^3}$ has consequently a self-conjugate sub-group of order 2.7^3 ur 4.7^3. Both these sub-groups have a self-conjugate G_{7^3}, which is also self-conjugate in $G_{8.7^3}$. This is contrary to the hypothesis. Consequently there is no group belonging here.

(*ii*) $p = 3$.

All the groups G_{216} contain 4 G_{27} which have a common sub-group G_9. This is a self-conjugate sub-group in G_{216}. The factor-group $G_{216}/G_9 = \Gamma_{24}$ which is formed by this has no self-conjugate sub-group G_3. In this case G_{216} would have a self-conjugate G_{27}, which conflicts with the assumption. There thus remain three different types for Γ_{24}. It follows directly from this that groups exist only when the conjugated sequence of Sylow's sub-groups consists of dihedral-, quaternion- or Abelian groups G_8 of type (1, 1, 1). When Γ_{24} is isomorphic with the octohedral group, G_{216} has a self-conjugate sub-group G_{108}, because Γ_{24} contains a tetrahedral group that is self-conjugate. Even in the case when Γ_{24} contains an Abelian G_8 of type (1, 1, 1), Γ_{24} has a self-conjugate tetrahedral group and thus G_{216} has a self-conjugate G_{108}. If, on the other hand, Γ_{24} contains a quaternion-group, G_{216} has a self-conjugate sub-group G_{72}, but may also simultaneously have a self-conjugate G_{108}. G_{72} contains 3 or 9 quaternion-groups and 1 G_9. If these G_8 have a common G_4, which is always the case when G_9 is cyclic, G_{216} contains a self-conjugate G_{108}, because the factor-group $G_{216}/G_4 = \Gamma_{54}$ has one of order 27. If, on the other hand, only the operation of order 2 is common, the factor-group Γ_{108} is formed. This has, as is proved below, either a self-conjugate G_4 or G_{27} and thus G_{216} contains a self-conjugate G_8 or G_{27}. The groups G_{216} sought thus have a self-conjugate sub-group G_{108}, except when the conjugate sequence of Sylow's sub-groups G_8 consists of 9 quaternion-groups without common operations. It is easy to show that in this case the self-conjugate sub-group G_{72} can only be chosen in one way. The group of isomorphisms of G_9 is a group of order 48. This contains 3 G_{16} with a common quaternion-group (P. 49). The isomorphims corresponding to the operations in G_8 are thus just these. This agrees of course with the preceding result (P. 26).

We may thus, for instance, conveniently choose G_{72} in the following way:

$$A^4 = B^4 = P_1^3 = P_2^3 = 1 \quad A^2 = B^2 \quad B^{-1}AB = A^3 \quad P_1P_2 = P_2P_1$$
$$A^{-1}P_1A = P_2^{-1} \quad B^{-1}P_1B = P_1^{-1}P_2^{-1}$$
$$A^{-1}P_2A = P_1 \quad B^{-1}P_2B = P_1^{-1}P_2$$

$G_9 = \{P_1, P_2\}$ is thus the common self-conjugate sub-group of 4 G_{27}. When these G_{27} are Abelian groups each operation in $\{P_1, P_2\}$ must be self-conjugate in a group G_{108} or G_{216}. We are thus brought back to the case where G_{216} has a self-conjugate G_{108}. Each G_{27} is self-conjugate in a sub-group G_{54}, which is generated by A^2 and the G_{27} given. G_{54} contains just 9 G_2, namely the operations of order 2 in the 9 quaternion-groups. An operation of order 3 must thus be permutable with A^2 and consequently G_{27} cannot contain an operation of order 9. There thus only remains the case that all the operations in G_{27} are of order 3. An operation of order 3, e. g. P_3 outside $\{P_1, P_2\}$, corresponds to an isomorphism of order 3 to $\{P_1, P_2\}$ and permutes the 9 quaternion-groups, i. e. A and B apart from operations in $\{P_1, P_2\}$. The group of isomorphisms G_{48} has a self-conjugate G_{24}, which in its turn contains 4 G_3. Thus G_{48} has 4 G_3 and consequently 8 isomorphisms of order 3. Any isomorphism of order 3 can be chosen corresponding to P_3, but then it is also established (1) which of the 4 G_3 in $\{P_1, P_2\}$ is central to $\{P_3, P_1, P_2\}$, (2) how P_3 permutes the operations A and B.

We may assume:

$$P_3^{-1}P_1P_3 = P_1 \quad P_3^{-1}P_2P_3 = P_1P_2$$

$\because$ $J_{P_3} = (A^2)\ (A\ \ B\ \ AB)\ (A^3\ \ A^2B\ \ A^3B)$ apart from operations in $\{P_1, P_2\}$.

P_3 permutes the 9 quaternion-groups in G_{72}

$$\because\ P_3^{-1}AP_3 = BU_1 \qquad P_3^{-1}BP_3 = ABU_2$$
$$\because\ P_3^{-2}AP_3^2 = ABU_2P_3^{-1}U_1P_3$$
$$\because\ P_3^{-3}AP_3^3 = BU_1ABU_2P_3^{-1}U_2P_3^{-1}U_1P_3^2$$
$$\because\ E = (AB)^{-1}U_1ABU_2P_3^{-1}U_2P_3P_3^{-2}U_1P_3^2 \quad . . . (11)$$

When this relation is satisfied it thus follows immediately that P_3^3 is permutable with B because

$$P_3^{-3}BP_3^3 = BU_1P_3^{-1}(AB)^{-1}U_1ABU_2P_3^{-1}U_2P_3^2$$

The different quaternion-groups are obtained by transforming $\{A, B\}$ with $P_1^xP_2^y$

$$(P_1^xP_2^y)^{-1}AP_1^xP_2^y = AP_1^{x-y}P_2^{x+y}$$

$$(P_1^xP_2^y)^{-1}BP_1^xP_2^y = BP_1^{2x+y}P_2^x$$

$$(P_1^xP_2^y)^{-1}ABP_1^xP_2^y = ABP_1^yP_2^{x+2y}$$

We may thus assume:

$U_1 = P_1^{2x+y}P_2^x \quad U_2 = P_1^yP_2^{x+2y}$ and because of (11) x and y can here have arbitrary values.

For $x = y = 0$ we obtain a group G_{216}, defined by the relations

$$A^4 = B^4 = P_1^3 = P_2^3 = P_3^3 = 1$$

$$A^2 = B^2 \quad A^{-1}BA = B^3 \quad P_1P_2 = P_2P_1 \quad P_1P_3 = P_3P_1$$

$$A^{-1}P_1A = P_2^{-1} \quad B^{-1}P_1B = P_1^{-1}P_2^{-1}$$

$$A^{-1}P_2A = P_1 \quad B^{-1}P_2B = P_1^{-1}P_2 \quad P_3^{-1}P_2P_3 = P_1P_2$$

$$P_3^{-1}AP_3 = B \quad P_3^{-1}BP_3 = AB$$

If one exchanges P_3 for $P_3P_1^xP_2^y$, all the other types are obtained from this.

We thus still have the case when G_{216} has a self-conjugate G_{108}. This G_{108} has 4 G_{27} which have a common G_9. The factor-group $G_{108}/G_9 = \Gamma_{12}$ has not a self-conjugate G_3 and is thus isomorphic with the tetrahedral group. As the isomorphic group to G_9 does not contain any tetrahedral group, each operation in a sub-group G_4 must be permutable with each operation in G_9. Thus G_{108} contains a non-cyclic G_4 self-conjugate. The conjugate sequence of Sylow's sub-groups thus consists of dihedral- or Abelian groups G_8 of the type (1, 1, 1).

G^1_{27}. $G_9 = \{P^3\}$.

(a) When G_8 is Abelian we may take

$$P^{-1}BP = C \quad P^{-1}CP = BC$$

$$\because A^{-1}PA = P^xU.$$

$$x^2 \equiv 1 \pmod{27}$$

Only $x \equiv 1$ gives an isomorphism. The groups are isomorphic and contain a self-conjugate G^3_8.

(b) When G_8 is a dihedral group we may take

$$P^{-1}A^2P = B \quad P^{-1}BP = A^2B$$

$$\because A^{-1}PA = P^xU.$$

Because $$A^{-2}PA^2 = PA^2B$$
it follows that

$$x \equiv -1 \qquad U = A^2 \text{ or } A^2B$$

Both lead to the same type, defined by the relations

$$A^4 = B^2 = P^{27} = 1 \quad B^{-1}AB = A^3$$

$$P^{-1}A^2P = B \quad P^{-1}BP = A^2B \quad A^{-1}PA = P^{-1}A^2B$$

G^2_{27}. $G_9 = \{P_1\}$ or $\{P_1{}^3, P_2\}$

(a) $$P_2{}^{-1}BP_2 = C \quad P_2{}^{-1}CP_2 = BC$$

$$\because A^{-1}P_1A = P_1{}^x \quad A^{-1}P_2A = P_1{}^{3x_1}P_2{}^{y_1}U$$

Since $\{B, C\}$ is self-conjugate in G_{216} and A^2 permutable with P_1 and P_2, we get

$$y_1 \equiv 1 \quad x^2 \equiv 1 \pmod 9 \quad x_1(x+1) \equiv 0 \pmod 3$$

For $x \equiv 1$ we may take U arbitrarily. The groups are isomorphic and contain a self-conjugate G^3_8.
For $x \equiv -1$ we may take both x_1 and U arbitrarily. The groups are isomorphic and give a single type G_{216} defined by the relations

$$A^2 = B^2 = C^2 = P_1^9 = P_2^3 = 1$$
$$AB = BA \quad AC = CA \quad BC = CB \quad P_1P_2 = P_2P_1$$
$$A^{-1}P_1A = P_1^{-1} \quad P_1B = BP_1 \quad P_1C = CP_1$$
$$AP_2 = P_2A \quad P_2^{-1}BP_2 = C \quad P_2^{-1}CP_2 = BC$$

(*b*) $G_9 = \{P_1\}$. $G_4 = \{A^2, B\}$

By an investigation analogous to (*a*), we obtain two types viz.

$$A^4 = B^2 = P_1^9 = P_2^3 = 1 \quad B^{-1}AB = A^3 \quad P_1P_2 = P_2P_1$$
$$P_2^{-1}A^2P_2 = B \quad P_2^{-1}BP_2 = A^2B \quad A^{-1}P_2A = P_2^{-1}A^2B$$
$$A^{-1}P_1A = P_1^{\pm 1} \quad B^{-1}P_1B = P_1$$

(*c*) $$P_1^{-1}BP_1 = C \quad P_1^{-1}CP_1 = BC$$
$$\because\ A^{-1}P_1A = P_1^xP_2^yU \quad A^{-1}P_2A = P_1^{3x_1}P_2^{y_1}$$

Hence it follows that

$$x^2 + 3x_1y \equiv 1 \quad 3x_1(x + y_1) \equiv 0 \pmod 9$$
$$y_1^2 \equiv 1 \quad y(x + y_1) \equiv 0 \pmod 3$$

Because $\{B, C\}$ is self-conjugate in G_{216} we have the following three cases:

$x \equiv 1 \pmod 9$

$$\because\ y_1^2 \equiv 1 \quad x_1y = y(x + y_1) \equiv x_1(x + y_1) \equiv 0 \pmod 3$$

For $y_1 \equiv 1$ the groups contain a self-conjugate G_8^3.

For $y_1 \equiv -1$ we obtain a single type G_{216} defined by the relations

$$A^2 = B^2 = C^2 = P_1^9 = P_2^3 = 1$$
$$AB = BA \quad AC = CA \quad BC = CB \quad P_1P_2 = P_2P_1$$
$$AP_1 = P_1A \quad P_1^{-1}BP_1 = C \quad P_1^{-1}CP_1 = BC$$
$$A^{-1}P_2A = P_2^{-1} \quad BP_2 = P_2B \quad CP_2 = P_2C$$

$x \equiv 4 \pmod 9$

$$\because\ y_1 \equiv -1 \text{ and } \begin{cases} x_1 \equiv 1 \\ y \equiv 1 \end{cases} \text{ or } \begin{cases} x_1 \equiv 2 \\ y \equiv 2 \end{cases} \pmod 3$$

$x \equiv 7 \pmod 9$

$$\because\ y_1 \equiv -1 \text{ and } \begin{cases} x_1 \equiv 1 \\ y \equiv 2 \end{cases} \text{ or } \begin{cases} x_1 \equiv 2 \\ y \equiv 1 \end{cases} \pmod 3$$

By exchanging A for AB, AC or ABC we may take $U = E$. New generating operations of G^2_{27} may be taken, e. g.

$$O_1 = P_1 P_2{}^a \qquad O_2 = P_1{}^3 P_2{}^b$$

so that $\qquad A^{-1}O_1A = O_1 \qquad A^{-1}O_2A = O_2{}^{-1}$

The four types are thus isomorphic with the immediately preceding.

(*d*) $G_9 = \{P_1{}^3,\ P_2\}$. $G_4 = \{A^2,\ B\}$

By an investigation analogous to (*c*), we obtain two types, viz.

$$A^4 = B^2 = P_1{}^9 = P_2{}^3 = 1 \quad B^{-1}AB = A^3 \quad P_1P_2 = P_2P_1$$

$$P_1{}^{-1}A^2P_1 = B \quad P_1{}^{-1}BP_1 = A^2B \quad A^{-1}P_1A = P_1{}^{-1}A^2B$$

$$A^{-1}P_2A = P_2{}^{\pm 1} \quad BP_2 = P_2B$$

G^3_{27}. $G_9 = \{P_1,\ P_2\}$.

(*a*) $\qquad P_3{}^{-1}BP_3 = C \quad P_3{}^{-1}CP_3 = BC$

A defines an isomorphism of order 2. We can then always choose new generating operations of $\{P_1,\ P_2\}$ so that

$$A^{-1}P_1A = P_1{}^x \quad A^{-1}P_2A = P_2{}^y \quad A^{-1}P_3A = P_1{}^{x_1}P_2{}^{y_1}P_3U$$

We obtain two types G_{216} defined by the relations

$$A^{-1}P_1A = P_1{}^{-1} \quad A^{-1}P_2A = P_2{}^{\pm 1} \quad A^{-1}P_3A = P_3$$

(*b*) $\qquad P_3{}^{-1}A^2P_3 = B \quad P_3{}^{-1}BP_3 = A^2B$

$$\because\ A^{-1}P_1A = P_1{}^x \quad A^{-1}P_2A = P_2{}^y \quad A^{-1}P_3A = P_1{}^{x_1}P_2{}^{y_1}P_3{}^{-1}U$$

We obtain three types G_{216} defined by the relations (except $x \equiv -y \equiv 1$)

$$A^{-1}P_1A = P_1{}^{\pm 1} \quad A^{-1}P_2A = P_2{}^{\pm 1} \quad A^{-1}P_3A = P_3{}^{-1}A^2B$$

G^6_{27}. $G_9 = \{P_1\}$ or $\{P_1^3, P_2\}$

(*a*)
$$P_2^{-1}BP_2 = C \quad P_2^{-1}CP_2 = BC$$
$$\because \; A^{-1}P_1A = P_1^x \quad A^{-1}P_2A = P_1^{3y}P_2U$$
$$\because \; x^2 \equiv 1 \pmod 9 \quad y(x+1) \equiv 0 \pmod 3)$$

For $x \equiv 1$ the groups contain a self-conjugate G^3_8.

For $x \equiv -1$ we get a single type G_{216} defined by the relations

$$A^2 = B^2 = C^2 = P_1^9 = P_2^3 = 1$$
$$AB = BA \quad AC = CA \quad BC = CB \quad P_1P_2 = P_2P_1^4$$
$$A^{-1}P_2A = P_2 \quad P_2^{-1}BP_2 = C \quad P_2^{-1}CP_2 = BC$$
$$A^{-1}P_1A = P_1^{-1} \quad P_1B = BP_1 \quad P_1C = CP_1$$

(*b*) $G_9 = \{P_1\}$. $G_4 = \{A^2, B\}$
When $\{P_1, P_2\}$ is transformed by A we must obtain
$$A^{-1}P_2A = P_1^{3y}P_2U$$
This is not possible, because $\{A^2, B\}$ is self-conjugate in G_{216}.

(*c*)
$$P_1^{-1}BP_1 = C \quad P_1^{-1}CP_1 = BC$$
$$\because \; A^{-1}P_1A = P_1^xP_2^yU \quad A^{-1}P_2A = P_1^{3x_1}P_2$$

Hence it follows that

$$x^2 + 3x_1y - \frac{3}{2}x^2y(x-1) \equiv 1 \pmod 9$$
$$y(x+1) \equiv x_1(x+1) \equiv 0 \pmod 3) \quad . \quad . \quad . \quad (12)$$
$$\because \; x \equiv 1 \quad y \equiv x_1 \equiv 0$$

The groups are isomorphic and contain a self-conjugate G^3_8.

(*d*)
$$P_1^{-1}A^2P_1 = B \quad P_1^{-1}BP_1 = A^2B$$
$$\because \; A^{-1}P_1A = P_1^xP_2^yU \quad A^{-1}P_2A = P_1^{3x_1}P_2$$

The congruences (12) must be satisfied and because $\{A^2, B\}$ is self-conjugate in G_{216} we get the following three cases:

$x \equiv -1 \pmod 9$

$$\because \; x_1 y + y \equiv 0 \pmod 3)$$

The generating operations of G^6_{27}

$$O_1 = P_1 P_2^a \quad O_2 = P_1^{3b} P_2$$

can always be taken so that

$$A^{-1} O_1 A = O_1^{-1} U \quad A^{-1} O_2 A = O_2$$

We thus obtain a single type G_{216} defined by the relations

$$A^4 = B^2 = P_1^9 = P_2^3 = 1 \quad B^{-1}AB = A^3 \quad P_1 P_2 = P_2 P_1^4$$
$$P_1^{-1} A^2 P_1 = B \quad P_1^{-1} B P_1 = A^2 B \quad A^{-1} P_1 A = P_1^{-1} A^2 B$$
$$A P_2 = P_2 A \quad B P_2 = P_2 B$$

$x \equiv 2$ or $5 \pmod 9$

In a way analogous to this we show that the four types obtained are isomorphic with the immediately preceding

G^7_{27}. $G_9 = \{P_1, P_2\}$

(*a*)
$$P_3^{-1} B P_3 = C \quad P_3^{-1} C P_3 = BC$$

We may take

$$A^{-1} P_1 A = P_1^x \quad A^{-1} P_2 A = P_2^y \quad A^{-1} P_3 A = P_1^{x_1} P_2^{y_1} P_3 U$$
$$\because \; x^2 \equiv y^2 \equiv 1 \quad x_1(x+1) \equiv y_1(y+1) \equiv 0 \bmod 3)$$

We get a single type G_{216} defined by the relations

$$A^{-1} P_1 A = P_1^{-1} \quad A^{-1} P_2 A = P_2^{-1} \quad A^{-1} P_3 A = P_3$$

(*b*) By an investigation analogous to (*a*), we obtain two types viz.

$$A^{-1} P_1 A = P_1^{\pm 1} \quad A^{-1} P_2 A = P_2^{\mp 1} \quad A^{-1} P_3 A = P_3^{-1} A^2 B$$

Finally I give a table showing the number of types for different values of p

p	Number of types
3	179
7	154
$p \equiv 1 \pmod 8$	239
$p \equiv 3 \pmod 4$	147
$p \equiv 5 \pmod 8$	195